Directed spaces: An extended framework for domain theory

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The theory of domains, which arose from logic and computer science and started with the works of Dona Scott in 1970's, is a foundation of the semantics of programming languages, logic and lambda calculus. After about 40 years of domain theory, one is forced to recognize that topology and domain theory have been beneficial to each other. As said in a book of Goubault-Larrecq, domain theory, in a broad sense, is topology done right.

The goal of this talk is to present an extended framework of domain theory through the topological method. The basic object introduced in this talk is a so-called directed space, which is a special T_0 space such that its topology is completely determined by the class of converging monotone nets (directed subsets) with respected to the specialization order. Many familiar structures, such as posets endowed with the Scott topologies, Alexandroff topological spaces, c-spaces, are directed spaces. The behavior of the category of directed spaces is very similar to the category of dcpos endowed with the Scott topologies.

Outline

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Basic notations we need

We do assume some knowledge of basic domain theory, topology and category theory:

- posets ($(P, \leq)$), directed subsets, directed complete posets (dcpo for short)
- upper set ($\uparrow A$, $\uparrow x$), lower set ($\downarrow A$, $\downarrow x$)
- Scott open set, the Scott topology ($\sigma(P)$), the upper topology ($\nu(P)$)
- monotone function, Scott continuous function
- T_0 space ($(X, O(X))$), net ($(x_j)_{j \in J}$) and limit ($(x_j) \to x$), ect.

Specialization order of a T_0 space

Given a T_0 space X, the partial order $\sqsubseteq$ defined on X by

$$x \sqsubseteq y \Leftrightarrow x \in \overline{\{y\}} \Leftrightarrow \mathcal{O}(x) \subseteq \mathcal{O}(y)$$

is called the specialization order, where $\overline{\{y\}}$ is the closure of $\{y\}$, $\mathcal{O}(x) = \{U \in O(X) : x \in U\}$.

From now on, all order-theoretical statements about T_0 spaces, such as upper sets, lower sets, directed sets, and so on, always refer to the specialization order "$\sqsubseteq$".

Directed open sets

Let $(X, O(X))$ be a T_0 space with the specialization order $\sqsubseteq$. Each directed subset D of X can regarded as a monotone net. We denote by $D \to x$ to mean that D is converging to x.

- For a directed $D \subseteq X$, $D \to x$ iff $D \cap U \neq \emptyset$ for all open neighborhood of x.

Set

$$DLim(X) = \{(D, x) : x \in X,\ D \text{ is a directed subset of } X \text{ and } D \to x\}$$

to be the set of all pairs of converging monotone nets and their limits in X.

- $\forall x, y \in X,\ x \sqsubseteq y \Leftrightarrow (\{y\}, x) \in DLim(X)$.

Definition

Let X be a T_0 space. A subset U of X is called directed open if for any $(D, x) \in DLim(X)$, $x \in U$ implies $D \cap U \neq \emptyset$.

Set

$$d(O(X)) = \{U \subseteq X : U \text{ is directed open}\}$$

to be the set of all directed open sets.

Theorem

Let X be a T_0 topological space. Then

(1) $O(X) \subseteq d(O(X))$;

(2) *For all $U \in d(O(X))$, $U = \uparrow U$;*

(3) $(X, d(O(X))$ *is a T_0 topological space with $\sqsubseteq_d = \sqsubseteq$, where $\sqsubseteq_d$ is the specialization order relative to $d(O(X))$.*

(4) $d(d(O(X)) = d(O(X))$.

The above result leads us to define a new topological space as follows.

Definition

A topological space X is said to be a directed space if it is T_0 and every directed open set is open; equivalently, $d(O(X)) = O(X)$. Particularly, the topology of a directed space X is denoted by $d(X)$.

One see that the idea to define a directed space is similar to define a sequential space and the Scott topology on a poset.

Theorem

Let X be a T_0 space, we have

(1) *X endowed with $d(O(X))$ is a directed space.*

(2) *The following conditions are equivalent to each other:*

  (i) *X is a directed space;*

  (ii) *For all $U \subseteq X$, U is open if and only if for any $(D, x) \in DLim(X)$, $x \in U$ implies $U \cap D \neq \emptyset$.*

  (iii) *For all $A \subseteq X$, A is closed if and only if for any directed subset $D \subseteq A$, $D \to x$ implies $x \in A$ for all $x \in X$.*

Examples of directed spaces

- Every poset endowed with the Alexandroff topology is a directed space.
- Every poset endowed with the Scott topology is a directed space.
- All c-spaces (resp. a-spaces) introduced by Erné (1981,1991) are directed spaces.
- A T_1 space is a directed space if and only if it is discrete.

Directed continuous functions

Definition

Let X, Y be two T_0 spaces. A function $f : X \longrightarrow Y$ is said to be directed continuous if it is monotone and preserves the limits of all directed subsets of X; equivalently, $(D, x) \in DLim(X)$ implies $(f(D), f(x)) \in DLim(Y)$.

Proposition

Given two T_0 spaces X, Y and a mapping $f : X \longrightarrow Y$, we have

(1) *f is directed continuous iff $f^{-1}(U) \in d(O(X))$ for any $U \in d(Y)$.*

(2) *If X, Y are two directed spaces, then f is continuous if and only if it is directed-continuous.*

The category **DTop** of directed spaces

In the following, we show that the category of directed spaces is a coreflective subcategory of the one of T_0 spaces.

Definition

We give two category as follows:

(1) The category $\mathbf{Top}_0$ is given as: objects are all nonempty T_0 topological spaces and morphisms are all continuous functions

(2) The category **DTop** is given as: objects are all nonempty directed spaces and morphisms are all (directed) continuous functions

Obviously, **DTop** is a full proper subcategory of $\mathbf{Top}_0$.

For a T_0 space X, we set

$$\mathcal{D}X = (X, d(O(X))).$$

Then $\mathcal{D}X$ is the directed space generated by directed open subsets of X. Given a continuous function $f : X \longrightarrow Y$ from X into another T_0 space Y, we set

$$f_D = f,$$

which is regarded as a mapping from $\mathcal{D}X$ into $\mathcal{D}Y$.

Proposition

f_D is (directed) continuous between $\mathcal{D}X$ and $\mathcal{D}Y$.

Theorem

Let X be a T_0 space. For any directed space Y and a map $f : Y \longrightarrow X$, f is continuous if and only if $f : Y \longrightarrow \mathcal{D}X$ is directed-continuous.

Corollary

DTop *is a coreflective full subcategory of* $\mathbf{Top}_0$.

In this section, we will see that a T_0 space can be endowed with several different directed topologies which generate the same specialization order.

We call them order compatible in the following. We will investigate the relationship among order compatible directed topologies on a poset. Particularly, we will show that the Scott topology on a dcpo is the coarsest compatible directed topology.

Definition

Let $(P, \leq)$ be a poset. A topology τ on P is called order compatible if the specialization order relative to τ agrees with the original one on P. Two T_0 topologies τ_1, τ_2 on a set X are called order compatible if they have the same specialization order.

- A topology τ on a poset P is order compatible if and only if $\nu(P) \subseteq \tau \subseteq \mathcal{A}(P)$.
- Let τ_1, τ_2 be two order compatible T_0 topologies on a set X. If $\tau_1 \subseteq \tau_2$, then $d(\tau_1)$ and $d(\tau_2)$ are order compatible directed topologies with $d(\tau_1) \subseteq d(\tau_2)$.
- Among all order compatible directed topologies on a post P, $d(\nu(P))$ is coarsest and $\mathcal{A}(P)$ is finest.

Particularly, we call the coarsest order compatible directed topology $d(\nu(P))$ the weak Scott topology and denote it by $\sigma_w(P)$.

It is easy to see that, the Scott topology on a poset is also order compatible.

What is the status of the Scott topology among all order compatible topoloies on a poset?

To answer this question, we need to define a new topology on a poset.

Let $(P, \leq)$ be a poset, $A \subseteq P$. Set $\mathrm{cl}_\sigma(A)$ to be the closure of A with respect to the Scott topology $\sigma(P)$. Let

$$\mathcal{B}_d = \{P \backslash \mathrm{cl}_\sigma(D) : D \subseteq P \ \text{ is directed}\},$$

then $\mathcal{B}_d \subseteq \sigma(P)$.

Definition

Let $(P, \leq)$ be poset. The topology on P generated by taken $\mathcal{B}_d$ as a subbase is called the strongly upper topology, denoted by $\nu_s(P)$.

We have the following properties.

- $\nu(P) \subseteq \nu_s(P) \subseteq \sigma(P)$. Hence, $\nu_s(P)$ is also a order compatible T_0 topology on P.
- $\nu(P) = \nu_s(P)$ if and only if for any directed subset $D \subseteq P$, either $cl_\sigma(D) = P$ or $cl_\sigma(D) = \bigcap\{\downarrow a : a \in P \ \&\ D \subseteq \downarrow a\}$. Hence, the upper topology coincides with the strongly upper topology for all dcpo.
- Generally, the upper topology is strictly coarser than the strongly upper topology.

Theorem

Let P be a poset. We have the follwing:

(1) $d(\nu(P)) = \sigma_w(P) \subseteq d(\nu_s(P)) = \sigma(P)$. *Hence, P endowed with the Scott topology $\sigma(P)$ is a directed space, and $\sigma(P)$ is the coarsest order compatible directed topology containing the strongly upper topology.*

(2) *The following two conditions are equivalent to each other:*

  (i) $\sigma_w(P) = \sigma(P)$;

  (ii) $\nu(P) = \nu_s(P)$.

This result indicates the exact position of the Scott topology on a dcpo as follows. Moreover, it also means that directed spaces is more general than posets endowed with the Scott topologies.

Corollary

The Scott topology on a dcpo is the coarsest order compatible directed topology.

In this section, we will show that the category **DTop** is cartesian closed, which behavior is very similar to the category of dcpos.

Binary products of **DTop**

Let X, Y be two directed spaces. Then the cartesian product $X \times Y$ has an partial order induced by the product topological space $X \times Y$ as follows: $\forall (x_1, y_1), (x_2, y_2) \in X \times Y$,

$$(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow x_1 \sqsubseteq x_2, \ \ y_1 \sqsubseteq y_2.$$

It is said to be the pointwise order on $X \times Y$.

Next, we define a topological space $X \otimes Y$ as follows:

- the underline set of $X \otimes Y$ is $X \times Y$;
- the topology on $X \otimes Y$ is generated in the following way: for any $\leq$-directed $D \subseteq X \times Y$ and for $(x, y) \in X \times Y$,

  $$D \to (x, y) \text{ in } X \otimes Y \Leftrightarrow \pi_1 D \to x \text{ in } X \text{ and } \pi_2 D \to y \text{ in } Y,$$

  that is, a subset $U \subseteq X \times Y$ is open if and only if $D \cap U \neq \emptyset$ for any $D \to (x, y)$ with $(x, y) \in U$.

Let $O(X \times Y)$ be the product topology of X and Y, let $O(X \otimes Y)$ be the topology defined as above. One always has $O(X \times Y) \subseteq O(X \otimes Y)$.

Theorem

For two directed spaces X and Y, we have

(1) $\mathcal{D}(X \times Y) = X \otimes Y$. *Hence, $X \otimes Y$ is a directed space such that its specialization order $\sqsubseteq$ is equal to the pointwise order $\leq$.*

(2) *$X \otimes Y$ is the binary product in* **DTop**.

(3) *Generally, the cartesian product topology $O(X \times Y)$ is not equal to $O(X \otimes Y)$.*

Hence, the category **DTop** has finite products. From now on, $X \otimes Y$ is called the directed product and the topology on $X \otimes Y$ is denoted by $d(X \otimes Y)$ for two directed spaces X and Y.

Proposition

Let X, Y, Z be three directed spaces. A function $f : X \otimes Y \longrightarrow Z$ is continuous if and only if it is continuous with respect to each variable.

Directed function spaces

Given two directed spaces X and Y, we set

$$Y^X = \{f : X \longrightarrow Y : \ f \text{ is continuous}\}$$

to be the set of all continuous functions from X into Y. There is a partial order $\leq$ on Y^X induced by X and Y as follows:

$$\forall f, g \in Y^X, \ f \leq g \Leftrightarrow f(x) \sqsubseteq g(x)$$

for all $x \in X$. We call this order the pointwise order on Y^X.

Next, we define a topological space $[X \to Y]$ as follows:

- the underline set of $[X \to Y]$ is Y^X;
- the topology on $[X \to Y]$ is generated in the following way: a subset $\mathcal{U} \subseteq Y^X$ is open if for any $\leq$-directed subset $\{f_i : i \in I\} \subseteq Y^X$ and for $f \in U$, $(f_i(x))_{i \in I} \to f(x)$ in Y for any $x \in X$ implies $U \cap \{f_i : i \in I\} \neq \emptyset$.

Theorem

For two directed spaces X and Y, $[X \to Y]$ is a directed space such that

(1) *Its specialization order $\sqsubseteq$ is equal to the pointwise order $\leq$,*

(2) *For a $\sqsubseteq$-directed subset $\{f_i : i \in I\} \subseteq Y^X$ and $f \in Y^X$,*

$$(f_i)_{i\in I} \to f \text{ in } [X \to Y] \Leftrightarrow (f_i(x))_{i\in I} \to f(x) \text{ in } Y$$

for all $x \in X$. For a subset $\mathcal{U}$ of $[X \to Y]$, $\mathcal{U}$ is open iff for a $\sqsubseteq$-directed subset $\{f_i : i \in I\} \subseteq Y^X$ with $(f_i)_{i\in I} \to f$, $f \in U$ implies $f_{i_0} \in U$ for some $i_0 \in I$.

From now on, $[X \to Y]$ is said to be the directed function space of the directed spaces X and Y, and the topology of $[X \to Y]$ is denoted by $d[X \to Y]$.

Cartesian closedness of of **DTop**

Given another directed space Z, there exist two natural bijective functions

$$\Lambda : Z^{X \times Y} \longrightarrow (Z^Y)^X$$

and

$$\Gamma : (Z^Y)^X \longrightarrow Z^{X \times Y}$$

defined as follows: $\forall f \in Z^{X \otimes Y}, \ \forall g \in (Z^Y)^X, \ \forall (x, y) \in X \times Y,$

$$\Lambda(f)(x)(y) = f(x, y), \ \ \Gamma(g)(x, y) = g(x)(y).$$

Always one has that

$$\Lambda \circ \Gamma(g) = g, \ \Gamma \circ \Lambda(f) = f.$$

Definition

Given directed spaces X, Y, the evaluation map $ev : [X \to Y] \otimes X \longrightarrow Y$ is defined as follows: $\forall f \in Y^X, \ \forall x \in X, \ ev(f, x) = f(x).$

- For directed spaces X, Y, the evaluation map $ev : [X \to Y] \otimes X \longrightarrow Y$ is continuous.

Theorem

Let X, Y, Z be three directed spaces. Then both Λ are Γ are isomorphisms with $\Gamma = \Lambda^{-1}$.

So we have obtained the following main result of this section.

Theorem

The category **DTop** *of all nonempty directed spaces is cartesian closed. Moreover, the binary product is $X \otimes Y$ and the exponential is $[X \to Y]$ for two directed spaces X and Y.*

In domain theory, the basic objects are dcpos or posets with the Scott topologies. Recall that **Dcpo** (resp. **Poset**) is the category of all dcpos (resp. posets) with Scott continuous functions. Particularly, **Dcpo** is cartesian closed. However, the properties of **Poset** is not as clear as that of **Dcpo**. In this section, we will compare the three categories **DTop**, **Dcpo** and **Poset** carefully, specially for their finite products and exponential objects. A counter-example is given to show that **Poset** is not cartesian closed.

As shown as above, each poset endowed with the Scott topology is a directed space. So both **Dcpo** and **Poset** can be regarded as full subcategories of **DTop**. Of course, **Dcpo** is also a full subcategory of **Poset**.

Let P, Q be two posets.

- The pointwise order on the cartesian product $P \times Q$ is defined as follows: $\forall (a, b), (x, y) \in P \times Q$, $(a, b) \leq (x, y) \Leftrightarrow a \leq x \;\&\; b \leq y$.
- For any directed subset $D \subseteq P \times Q$, if $\bigvee \pi_1 D$ and $\bigvee \pi_2 D$ exist, then $\bigvee D = (\bigvee \pi_1 D, \bigvee \pi_2 D)$.
- Set

$$[P \rightarrow_s Q] = \{f : P \rightarrow Q \mid f \text{ is a Scott continuous function}\}$$

endowed with the pointwise order: $\forall f, g \in [P \rightarrow_s Q]$, $f \leq g \Leftrightarrow \forall x \in P,\ f(x) \leq g(x)$. If Q is a dcpo, then $([P \rightarrow_s Q], \leq)$ is a dcpo and for any directed subset $S \subseteq [P \rightarrow_s Q]$, $\forall x \in P,\ (\bigvee S)(x) = \bigvee_{s \in S} s(x)$.

From now on, $[P \rightarrow_s Q]$ is always endowed with the pointwise order and the Scott topology, and we call $[P \rightarrow_s Q]$ the Scott function space. Since P and Q endowed with the Scott topologies are directed spaces, we remark specially that the notation $[P \rightarrow Q]$ denotes the directed function space in the category **DTop** of directed spaces, i.e., the topology on $[P \rightarrow Q]$ is $d[P \rightarrow Q]$ defined in the last section.

Comparing **DTop** with **Dcpo** and **Poset**

Proposition

For two posets P and Q endowed with the Scott topologies, $P \otimes Q = (P \times Q, \sigma(P \times Q))$.

Next, we consider the function spaces in **DTop**, **Dcpo** and **Poset**.

Definition

Let P, Q be two posets, $S \subseteq [P \to_s Q]$ is a directed subset with an existing sup $\bigvee S$ in $[P \to_s Q]$. We call the sup $\bigvee S$ pointwise if for any $x \in P$, $\bigvee_{s \in S} s(x)$ exists in Q and $(\bigvee S)(x) = \bigvee_{s \in S} s(x)$.

Proposition

Let P, Q be two posets endowed with the Scott topologies. The following two conditions are equivalent to each other:

(1) $d[P \to Q] \subseteq \sigma([P \to_s Q])$.

(2) *All existing directed sups in* $[P \to_s Q]$ *are pointwise.*

Moreover, if Q is a dcpo, then $d[P \to Q] = \sigma([P \to_s Q])$, that is, $[P \to Q] = [P \to_s Q]$

Hence, for two dcpos P and Q, the product $P \times Q$ and the Scott function space $[P \to_s Q]$ agree with those in the category **DTop** of directed spaces. So the following result holds.

Theorem

The embedding functor $\mathcal{E}$: **Dcpo** $\longrightarrow$ **DTop** *preserves finite products and exponential objects.*

A counter-example for the case of posets

Next, we consider the category **Poset**.

Let $P = \mathbb{N} \cup \{\infty_1, \infty_2\}$, where $\mathbb{N}$ is the set of all natural numbers and $\infty_1 \neq \infty_2$. An order on P is defined as follows: $\forall x, y \in P$, $x \leq y$ iff one of the following conditions holds:

- $x \in \mathbb{N}$ and $y \in \{\infty_1, \infty_2\}$,
- $x, y \in \mathbb{N}$ and $x \leq y$,
- $x = y = \infty_1$ or $x = y = \infty_2$.

Then $(P, \leq)$ is a poset such that the chain $\mathbb{N}$ has two minimal upper bounds.

Consider the function space $[2 \to_s P]$, where $2 = \{0, 1\}$ with $0 < 1$. For each $n \in \mathbb{N}$, we define a map $f_n : 2 \to P$ as follows:

$$f_n(0) = n, \ \ f_n(1) = \infty_1.$$

Then $f_n \in [2 \to_s P]$ and $f_n \leq f_m$ whenever $n \leq m$. So $\{f_n : n \in \mathbb{N}\}$ is directed.

One see that $\bigvee_{n \in \mathbb{N}} f_n = f_{\infty_1}$, where $f_{\infty_1}(0) = f_{\infty_1}(1) = \infty_1$. However, note that $\{f_n(0) : n \in \mathbb{N}\} = \mathbb{N}$ and $\mathbb{N}$ doesn't have a least upper bound, but

$$(\bigvee_{n \in \mathbb{N}} f_n)(0) = f_{\infty_1}(0) = \infty_1.$$

It means that the suprema $\bigvee_{n \in \mathbb{N}} f_n$ in $[2 \to_s P]$ is not pointwise, i.e.,

$$d[2 \to P] \not\subseteq \sigma([2 \to_s P]).$$

For any two posets P, Q in the category **Poset**, the underline sets of $[P \to_s Q]$ and $[P \to Q]$ are both equal to the set of all Scott continuous functions from P into Q. But, as shown by the above counter-example, the following equation

$$[P \to_s Q] = [P \to Q]$$

doesn't hold generally.

It had been proved by A. Jung (1989) that, a full subcategory of **Dcpo** is cartesian closed if and only if it is closed under finite (cartesian) products and the Scott function spaces. For the category **Poset**, it must add an extra condition to characterise cartesian closed full subcategories as follows.

The conditions for cartesian closedness of posets

Theorem

A full subcategory $\mathbf{A}$ *of* **Poset** *is cartesian closed if and only if the following conditions hold:*

(1) $\mathbf{A}$ *is closed under finite (cartesian) products and the Scott function spaces,*

(2) *For all* $A, B \in \mathrm{Ob}(\mathbf{A})$, $d[A \to B] \subseteq \sigma([A \to_s B])$*; equivalently, all existing directed sups in* $[P \to_s Q]$ *are pointwise.*

Combining the counter-example and the above result, we have the following result.

Corollary

Poset *is not cartesian closed.*

As shown in the last section, the category **DTop** of directed spaces is cartesian closed. Obviously, the following result holds.

Proposition

A full subcategory **A** *of* **DTop** *is cartesian closed if it is closed under finite directed products and directed function spaces.*

Does the reverse of the above proposition hold?

We only have the following partial answer.

Proposition

If **A** *is a cartesian closed full subcategory of* **DTop**, *then the finite products in* **A** *coincide with those in* **DTop**.

We do not know whether the exponential objects in **A** coincide with the directed function spaces in **DTop**.

It is more possible that the answer is negative that, one find a cartesian closed full subcategory **A** of **Poset** such that **A** contains two objects P, Q satisfying $\sigma([P \to Q]) \nsubseteq d([P \to_s Q])$.

We have seen that directed spaces are very similar to dcpos or posets endowed with the Scott topology. In this section, we will introduce a so-called continuous space in a very natural way. The notion of continuity in domain theory can be smoothly extend to T_0 spaces, especially to directed spaces.

Approximating relation on directed spaces

Definition

Let X be a directed space, $x, y \in X$. We say that x is d-approximating y, denoted by $x \ll_d y$, if for any directed subset $D \subseteq X$, $D \to y$ implies $x \sqsubseteq d$ for some $d \in D$. If $x \ll_d x$, then x is called a d-compact element of X.

We introduce the following notation for $x, y \in X$ and $A \subseteq X$:

$$\begin{aligned} x \ll_d y \quad &\Leftrightarrow \quad x \text{ is } d - \text{approximating } y \\ \mathord{\downarrow\!\!\!\!\downarrow}_d x \quad &= \quad \{y \in X : y \ll_d x\} \\ \mathord{\uparrow\!\!\!\!\uparrow}_d x \quad &= \quad \{y \in X : x \ll_d y\} \\ K_d(X) \quad &= \quad \{x \in X : x \ll_d x\} \end{aligned}$$

Based on the d-approximating relation, a notion of a d-continuous space is defined as follows.

Definition

A topological space X is called d-continuous if it is a directed space such that $\downarrow\!\!\!\downarrow_d x$ is directed and $\downarrow\!\!\!\downarrow_d x \to x$ for all $x \in X$.

One see that, when P is a poset endowed with the Scott topology $\sigma(P)$, P is d-continuous if and only if P is continuous in the sense of the classical domain theory.

Lemma

Let X be a d-continuous space. Then we have the following:

(1) *For all $x, y \in X$, $x \ll_d y$ implies $x \ll_d z \ll_d y$ for some $z \in X$.*

(2) $\Uparrow_d x = (\uparrow x)^\circ$ *for all $x \in X$. Moreover, $\{\Uparrow_d x : x \in X\}$ is a basis of the topology of X.*

(3) *For all $x, y \in X$, the following are equivalent:*

   (i) $x \ll_d y$;

   (ii) $y \in (\uparrow x)^\circ$.

   (iii) *For any net $(x_j)_J \subseteq X$, $(x_j) \to y$ implies $x \sqsubseteq x_{j_0}$ for some $j_0 \in J$.*

Approximating relation on T_0 spaces

From statement (iii) of the above lemma, we can define a new approximating relation on any T_0 space as follows.

Definition

Let x, y be elements of a T_0 space X. We say that x *n*-approximates y if for all net $\{y_j\}_{j\in J}$ (shortly, $\{y_j\}_J$) of X, $(y_j) \to y$ implies $x \sqsubseteq y_{j_0}$ for some $j_0 \in J$. We say that x is n-compact if it n-approximates itself.

We introduce the following notations for $x, y \in X$ and $A \subseteq X$:

$$\begin{aligned} x \ll_n y &\Leftrightarrow x\ n-\text{approximates}\ y \\ \Downarrow_n x &= \{y \in X : y \ll_n x\} \\ \Uparrow_n x &= \{y \in X : x \ll_n y\} \\ K_n(X) &= \{x \in X : x \ll_n x\} \end{aligned}$$

Since the *n*-approximation does not depend on directed topologies, a notion of *n*-continuity for a T_0 space can be defined as follows.

Definition

A topological space X is called *n-continuous* if X is T_0 and for all $x \in X$, there exists a net $\{x_j\}_J \subseteq \,\downarrow\!\!\!\!\downarrow x$ such that $(x_j) \to x$.

For the concept of general continuity in domain theory, the notion of a c-space is another famous generalization.

Definition

(Erné 1981,1991, Ershov 1993,1997) A topological sapce X is called a c-space, if it is T_0 and every point y has a neighborhood basis of sets of the form $\uparrow x$.

Together with the notions of d-continuity and n-continuity introduced above, there are three generalizations of continuity in domain theory. What is the relationship among them?

d-continuous = n-continuous = c-space

Theorem

For a topological space X, the following are equivalent:

(1) X *is d-continuous;*

(2) X *is a directed space and for each* $x \in X$, *there is a directed subset* $D \subseteq \downarrow_d x$ *such that* $D \to x$.

(3) X *is n-continuous;*

(4) X *is a* T_0 *space and for each* $x \in X$, $\downarrow_n x$ *is directed and converges to* x;

(5) X *is a* T_0 *space and for each* $x \in X$, *there is a directed subset* $D \subseteq \downarrow_n x$ *such that* $D \to x$;

(6) X *is a c-space.*

(7) X *is* T_0 *and* $O(X)$ *is a completely distributive lattice.*

A uniform name of continuity

From now on, we give a uniform name to replace the three notions of continuity.

Definition

A topological space is called continuous if one of the seven equivalent conditions in the above theorem holds.

(1) For not leading to confusion, a continuous dcpo or poset always means the order-theoretic one in domain theory.

(2) Every continuous poset is exactly a continuous space endowed with the Scott topology.

(3) Every T_0 Alexandroff space is a continuous space.

(4) Every continuous space is a directed space.

(5) Similarly, we can define a notion of an algebraic space.

Category **CTop** of continuous spaces

Definition

The category **CTop** is given by:

(1) Objects are all continuous spaces,

(2) Morphisms are all (directed) continuous functions.

So, **CTop** is a full subcategory of **DTop**. Moreover, we have

Theorem

CDcpo $\subseteq$ **CPoset** $\subseteq$ **CTop** $\subseteq$ **DTop**, *where* **CDcpo** *(resp.* **CPoset***) is the category of continuous dcpo (resp. posets) with Scott continuous functions.*

Since **DTop** is cartesian closed and **CTop** $\subseteq$ **DTop**, an interesting problem is:

Classify the category **CTop** by cartesian closedness as continuous domains in domain theory.

We introduce a new topological space, called a directed space, which is a special T_0 space such that its topology is completely determined by the class of converging monotone nets (directed subsets).

(1) **DTop** is cartesian closed and is a co-reflective full subcategory of $\mathbf{Top_0}$;

(2) All posets endowed with the Scott topologies are directed spaces. Specially, the Scott topology is the coarsest order compatible directed topology on a dcpo;

(3) A continuous (resp. an algebraic) space is introduced through a new approximating relation on T_0 space, which is a special directed space and is equal to a *c*-space (resp. an *a*-space) investigated by Ernè firstly.

These results lead that we can develop an extended framework of domain theory of this kind almost as smoothly as the usual domain theory.

Thank you!