Join-continuity + Hypercontinuity = Prime-continuity

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Outline

Meet-continuity

A meet-semilattice P is *meet-continuous* if

(1) it is a dcpo; and

(2) for all $x \in P$ and all directed subsets D of P, it holds that

$$x \wedge \bigvee D = \bigvee (x \wedge D),$$

where $x \wedge D := \{x \wedge d \mid d \in D\}$.

Meet-continuity

Theorem (Theorem O-4.2,[1])

In a directed complete semilattice L, the following are equivalent:

(1) L *is meet-continuous;*

(2) *for each* $x \in L$ *and any directed subset* D *of* P*, whenever* $x \leq \bigvee D$ *then*
$$x = \bigvee (x \wedge D).$$

(3) $(x, y) \mapsto x \wedge y : L \times L \longrightarrow L$ *preserves directed sups;*

(4) *for each* $x \in L$ *and each directed net* $(x_j)_{j \in J}$*, we have*
$$x \wedge \bigvee_{j \in J} x_j = \bigvee_{j \in J} (x \wedge x_j).$$

Meet-continuity

Theorem (Continued)

If L is a lattice, then these conditions are also equivalent to:

(8) *for each $x \in L$ and any family $(x_j)_{j \in J}$, we have*

$$x \wedge \bigvee_{j \in J} x_j = \bigvee_{A \in \mathrm{fin}(J)} \left(x \wedge \bigvee_{j \in A} x_j \right),$$

where $\mathrm{fin}(J)$ is the set of all finite subsets of J.

Meet-continuity

Let L be a lattice. Then the following are equivalent:

(1) L is a *frame*, i.e., for each $x \in L$ and any family $(x_j)_{j \in J}$, we have

$$x \wedge \bigvee_{j \in J} x_j = \bigvee_{j \in J} (x \wedge x_j).$$

(2) L is meet-continuous and distributive.

Meet-continuity

Theorem ([1])

The following statements are equivalent for a directed complete meet-semilattice L:

(1) L *is meet-continuous;*

(2) *for each* $x \in L$ *and any directed subset* D *of* L*, whenever* $x \leq \bigvee D$ *then*

$$x \in \mathrm{cl}_\sigma(\downarrow x \cap \downarrow D),$$

where cl_σ *is the Scott closure operator.*

Meet-continuity

Definition (Meet-continuous poset)

A poset P is said to be *meet-continuous* if for any $x \in P$ and any directed subset D whose supremum exists, whenever $x \leq \bigvee D$ then

$$x \in \mathrm{cl}_\sigma(\downarrow x \cap \downarrow D).$$

A remarkable result by Kou, Liu & Luo

For a dcpo ...

Continuity = Quasicontinuity + Meet-continuity.

A remarkable result by Kou, Liu & Luo

For a dcpo ...

Continuity = Quasicontinuity + Meet-continuity.

... first reported in [2], but contained a gap.

Continuity

Definition (Continuous poset)

A poset P is called *continuous* if for all $y \in P$, we have

$$y = \bigvee \{x \in P \mid x \ll y\},$$

where $x \ll y$ if whenever a directed subset D whose $\bigvee D$ exists and $\bigvee D \geq y$ then $x \in \downarrow D$.

Quasicontinuity

Definition (Quasicontinuous poset)

A poset P is called *quasicontinuous* if the poset of nonempty finitely generated upper sets, $(\mathcal{U}_f(P), \supseteq)$, is continuous.

A sequence of technical lemmas

To fill the gap in their proof, [1] used a sequence of technical lemmas to establish the result:

Proposition III-3.6(ii) $\longrightarrow$ Lemma III-2.10 $\longrightarrow$ Lemma III-3.3 $\longrightarrow \ldots$

The result is reported in the form of Propositions III.-2.4 and III.3-10.

A sequence of technical lemmas

Proposition (Prop. III-3.6(ii),[1])

Let S be a quasicontinuous domain.
For any $\varnothing \neq H$ in S,

$$\Uparrow H = \mathrm{int}_\sigma(\uparrow H).$$

A sequence of technical lemmas

Lemma (Lem. III-2.10, [1])

If F is a finite set in a meet continuous dcpo, then we have

$$\mathrm{int}_\sigma(\uparrow F) \subseteq \bigcup\{\uparrow x : x \in F\}.$$

A sequence of technical lemmas

Lemma (Rudin's Lemma: Lem. III-3.3, [1])

Let $\mathcal{F}$ be a directed family of nonempty finite subsets of a poset P. Then there exists a directed set $D \subseteq \bigcup_{F \in \mathcal{F}} F$ such that $D \cap F \neq \emptyset$ for all $F \in \mathcal{F}$.

A closer look via Stone dual

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Prime-Continuity

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Prime-Continuity = Hypercontinuity

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For a distributive complete lattice ...

Prime-Continuity = Hypercontinuity + ?-continuity.

A reassuring fact

Proposition (Prop. II-4.17, [1])

Let L be a complete lattice. The following conditions are equivalent:

(1) L *is meet continuous (O-4.1);*

(2) $\sigma(L)$ *is join continuous (O-4.1);*

(3) $\sigma(L)^{\mathrm{op}}$ *is a frame.*

A reassuring fact

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Here, join-continuity is just the dual notion of meet-continuity.

Back to the big picture

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For a distributive complete lattice ...

Prime-Continuity = Hypercontinuity + Join-continuity.

What we hope ...

Conjecture

The following statements are equivalent for a dcpo P:

(1) P is meet-continuous.

(2) $\sigma(P)$ is join-continuous.

(3) $\sigma^{\mathrm{op}}(P)$ is a frame.

Serendipity

Theorem (Theorem 3.8,[1])

The following statements are equivalent for a poset P:

(1) P *is meet-continuous.*

(2) $\sigma(P)$ *is join-continuous.*

(3) $\sigma^{\mathrm{op}}(P)$ *is a frame.*

Maybe we can achieve more

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Surer than sure

Theorem ([2])

The following statements are equivalent for a poset P:

(1) *P is a continuous poset.*

(2) *$\sigma(P)$ is a prime-continuous lattice.*

Surer than sure

Theorem ([2])

The following statements are equivalent for a poset P*:*

(1) P *is a continuous poset.*

(2) $\sigma(P)$ *is a prime-continuous lattice.*

Theorem ([3])

The following are equivalent for a poset P*:*

(1) P *is a quasicontinuous poset.*

(2) $\sigma(P)$ *is a hypercontinuous lattice.*

Main result

Theorem

The following statements are equivalent for a distributive complete lattice L:

(1) L *is join-continuous and hypercontinuous.*

(2) L *is prime-continuous.*

Join-continuous lattice

A distributive complete lattice L is join-continuous if for all $x \in L$ and all $S \subseteq L$, we have

$$x \vee \bigwedge S = \bigwedge \{x \vee s \in s \in S\}.$$

Hypercontinuous lattice

Definition (Hypercontinuity)

A complete lattice L is called *hypercontinuous* if for all $y \in L$, we have

$$y = \bigvee \{x \in L | x \prec y\},$$

where $x \prec y$ if whenever the intersection of a nonempty collection of upper sets is contained in $\uparrow y$, then the intersection of finitely many is contained in $\uparrow x$.

Prime-continuous lattice

Definition (Prime-continuity)

A complete lattice L is said to be *prime-continuous* if for all $y \in L$, we have

$$y = \bigvee \{x \in L | x \lll y\},$$

where $x \lll y$ if for all $S \subseteq L$, whenever $\bigvee S \geq y$ then $x \in \downarrow S$.

Completely distributive lattice

Definition (Complete distributivity)

A complete lattice L is said to be *completely distributive* if for all families $(u^i_j)_{j \in J}$, one for each $i \in I$,

$$\bigwedge_{i \in I} \bigvee_{j \in J_i} u^i_j = \bigvee_{f \in \prod_{i \in I} J_i} \bigwedge_{i \in I} u^i_{f(i)}.$$

Raney's characterisation of CDL

Theorem ([2])

A complete lattice is completely distributive iff it is prime continuous.

Raney's characterisation of CDL

Theorem ([2])

A complete lattice is completely distributive iff it is prime continuous.

complete distributivity = prime-continuity ...

$$\text{prime-continuity} \implies \text{join-continuity} + \text{distributive}$$

and

$$\text{prime-continuity} \implies \text{frame}$$

Sup-inf characterisations

Theorem

Let L be a complete lattice.

(1) *L is continuous iff for all $x \in L$,*

$$x = \bigvee \{ \bigwedge U \mid x \in U \in \sigma(L) \}.$$

(2) *L is hypercontinuous iff for all $x \in L$,*

$$x = \bigvee \{ \bigwedge (L \setminus \downarrow M) \mid M \text{ is a finite subset of } L,\ x \notin \downarrow M \}.$$

(3) *L is prime-continuous iff for all $x \in L$,*

$$x = \bigvee \{ \bigwedge (L \setminus \downarrow y) \mid x \notin \downarrow y \}.$$

Chain of implications

As a result...

prime-continuity $\Longrightarrow$ hypercontinuity $\Longrightarrow$ continuity

Main result (one direction)

Theorem

The following statements are equivalent for a distributive complete lattice L:

(1) L *is join-continuous and hypercontinuous.*

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Main result (one direction)

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(1) L *is join-continuous and hypercontinuous.*

(2) L *is prime-continuous.*

(2) $\implies$ (1).

Prime continuity implies join continuity and hypercontinuity. □

Main result (the converse)

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Main result (the converse)

(1) $\implies$ (2)

By the sup-inf characterisation of prime-continuity, we must show that for any $x \in L$,

$$x = \bigvee \{ \bigwedge (L \backslash \downarrow y) \mid x \notin \downarrow y \}.$$

But by the sup-inf characterisation of hypercontinuity, we have

$$x = \bigvee \{ \bigwedge (L \backslash \downarrow M) \mid M \text{ is a finite subset of } P,\ x \notin \downarrow M \}.$$

□

Key idea

Given that L is join-continuous ...

$$\bigwedge(L \setminus \downarrow\{m_1, \ldots, m_n\}) = \bigvee_{k=1}^{n} \bigwedge(L \setminus \downarrow m_k).$$

Lemma

Lemma

Let L *be a join-continuous complete lattice. Then for any non-empty finite set* $M = \{m_1, \ldots, m_n\} \subseteq L$*, the following equation holds:*

$$\bigwedge(L \backslash \downarrow M) = \bigvee_{k=1}^{n} \bigwedge(L \backslash \downarrow m_k).$$

Proof by induction on n

Base case: $n = 1$
The equation is trivially true.

Proof by induction on n

Inductive hypothesis:

$$\bigvee_{k=1}^{n} \bigwedge (L \backslash \downarrow m_k) = \bigwedge (L \backslash \downarrow \{m_1, \ldots, m_n\}).$$

Proof by induction on n

Inductive step:

$$\bigvee_{k=1}^{n+1} \bigwedge (L \backslash \downarrow m_k) = (\bigwedge (L \backslash \downarrow m_{n+1})) \vee \bigwedge (L \backslash \downarrow \{m_1, \ldots, m_n\})$$

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$$\overset{j.c.}{=} \bigwedge \{(\bigwedge (L \backslash \downarrow m_{n+1})) \vee s \mid s \in \bigcap_{i=1}^{n} (L \backslash \downarrow m_i)\}$$

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$$= \bigwedge \underbrace{\left\{ r \vee s \mid r \in (L \setminus \downarrow m_{n+1}) \text{ and } s \in \bigcap_{i=1}^{n} (L \setminus \downarrow m_i) \right\}}_{X}$$

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$$\overset{X=Y}{=} \bigwedge \underbrace{(L \backslash \downarrow \{m_1, \ldots, m_n, m_{n+1}\})}_{Y}$$

Concluding remarks

We have given an (alternative) proof of the result:

For a poset ...

Continuity = Quasi-continuity + Meet-continuity

by considering the Stone duals, i.e., lattice of Scott opens, and the following new result:

For a distributive complete lattice ...

Prime-continuity = Hypercontinuity + Join-continuity

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The gap explained

Note that $\sigma(P)$ is a hypercontinuous lattice and $\Gamma(P)$, the lattice of Scott closed sets, is a generalized continuous lattice when P is quasi-continuous [4]. So if P is also meet-continuous, then from Theorem 2.6, $\Gamma(P)$ is continuous. Therefore, $\sigma(P)$ is a completely distributive lattice. Note that since (P, σ) is isomorphic to the spectrum of $\sigma(P)$, then we have

Theorem 2.5. *A quasicontinuous dcpo P is continuous if and only if it is a mc-dcpo.*

Notice in the above argument, if in addition P is meet continuous, then $\Gamma(P)$ is continuous. The justification for this uses Theorem 2.6 (which cannot be found in the paper). By using the meet continuity alone, speculating the content of Theorem 2.6, there seems no way one can deduce the continuity of $\Gamma(P)$. Indeed one there are examples of meet-continuous dcpo whose lattice of Scott opens is not continuous. In our current work, we patch this gap in Kou's argument by augmenting hypercontinuity with join-continuity.