Combining the Hoare powerdomain with the probabilistic powerdomain

Klaus Keimel [1] & Gordon D. Plotkin [2]

[1]Faculty of Mathematics, Technical University of Darmstadt

[2]Laboratory for the Foundations of Computer Science, School of Informatics, University of Edinburgh

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Introduction

- Consider mixed powerdomains for probability and ordinary nondeterminism (after [TKP], [KP]).
- Expect three, corresponding to the three kinds of domain-theoretic nondeterminism: lower (Hoare), upper (Smyth), and convex (Plotkin) [Mis]
- We take an algebraic point of view, emphasising (in)equational axioms,
- particularly accepting that probabilistic choice distributes over nondeterministic choice:

$$x +_r (y \cup z) = (x +_r y) \cup (x +_r z) \qquad (r \in [0, 1])$$

- Other domain-theoretic authors following this path: Oxford PRG [1999], Yang, Mislove, Tix, Goubault-Larrecq.
- In each of the three cases we obtain three results: free algebra characterisations, functional representations [G-L], and healthy predicate transformers.
- In this talk only the lower — Hoare — case is considered: it is the simplest, and illustrates the main themes.

Method

- We discuss the 'ingredients', the Hoare and probabilistic powerdomains, before the 'dish', the Hoare mixed powerdomain.
- We proceed abstractly (function-analytically) considering general structures — Kegelspitzen — and deducing the particular case. (Only needed in Hoare case for last two results.)
- To obtain results for Kegelspitzen we use previous results for d-cones.
- If time permits we will discuss the other distributive law:

$$x \cup (y +_r z) = (x \cup y) +_r (x \cup z) \qquad (r \in [0, 1])$$

Algebraic context

- Our algebras are dcpos with finitary Scott-continuous operations.
- These operations may also continuously depend on parameters varying over a fixed dcpo, for example, $\mathbb{I} =_{\text{def}} [0, 1]$ or $\overline{\mathbb{R}}_+$.
- Homorphisms are continuous and preserve the operations (parametrically).
- We consider classes of such algebras given by inequational axioms $t \leq u$, with the parameters set to constants.
- Free algebras always exist, and the map $f \mapsto \overline{f}$ extending a function to a homomorphism is itself continuous.

Axioms for the Hoare (lower) powerdomain

Semilattice

$$\left\{ \begin{array}{rcl} (x \cup y) \cup z & = & x \cup (y \cup z) \\ x \cup y & = & y \cup x \\ x \cup x & = & x \end{array} \right.$$

Join

$$x \leq x \cup y$$

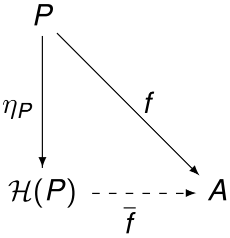
Bottom

$$\bot \leq x$$

Such algebras are the complete semilattices with $\cup = \vee$ and with $\bot$ the least element.

The Hoare powerdomain $\mathcal{H}(P)$

- $\mathcal{H}(P)$ consists of the Scott-closed subsets of P ordered by inclusion, and with $\cup$ being set-theoretic union and $\perp$ being the empty set.
- It is the free join-semilattice with a least element over P:



where

$$\eta_P(x) = \downarrow x \quad \text{and} \quad \bar{f}(C) = \bigvee_{x \in X} f(x)$$

- $\mathcal{H}(P)$ is a domain if P is.

Axioms for probabilistic choice

Barycentric algebra Has operations $+_r$ ($r \in [0,1]$) such that

$$\begin{cases} x +_1 y & = & x \\ x +_r x & = & x \\ x +_r y & = & y +_{1-r} x \\ (x +_p y) +_r z & = & x +_{pr} (y +_{\frac{r(1-p)}{1-pr}} z) \quad \text{(provided } pr < 1\text{)} \end{cases}$$

Convex space Has affine sum operations such that

$$\begin{cases} \sum_{i=1}^n \delta_{ij} x_i & = & x_j \\ \\ \sum_{i=1}^n p_i \big(\sum_{j=1}^m q_{ij} x_j \big) & = & \sum_{j=1}^m \big(\sum_{i=1}^n p_i q_{ij} \big) x_j \end{cases}$$

where δ_{ij} is the Kronecker symbol

Kegelspitzen (Cone tips)

A Kegelspitze is a Scott-continuous barycentric algebra with a constant 0 and an action $\cdot : \mathbb{I} \times A \to A$ continuously parameterised on $\mathbb{I}$ such that:

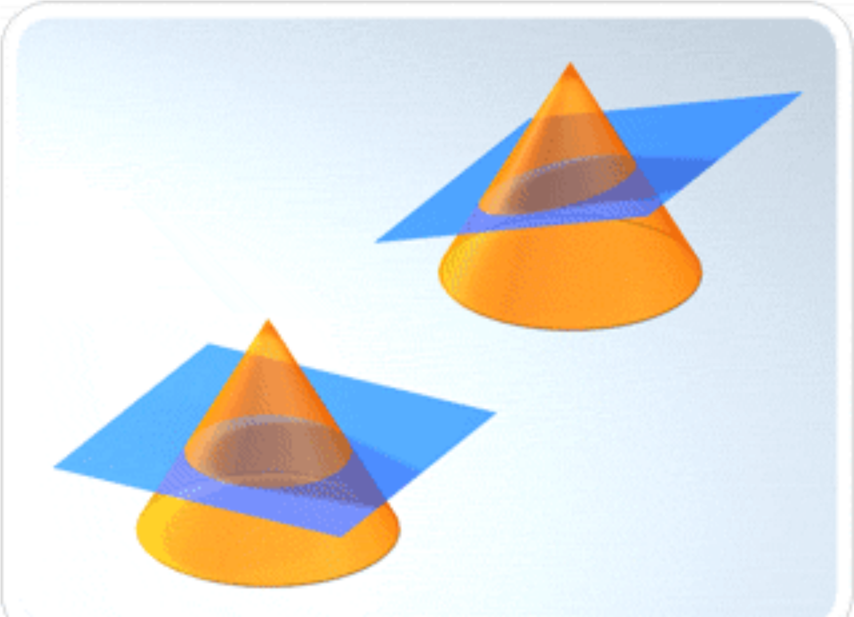
$$r \cdot x = x +_r 0$$

(Then $0 = \perp$ as $0 = 0 \cdot x \leq 1 \cdot x = x$.)

Example $\mathbb{I}$ where $x +_r y = rx + (1 - r)y$ and $0 = 0$.

Kegelspitze homomorphisms are called linear.
They are the Scott-continuous functions which are affine (i.e., which preserve the $+_r$) and which preserve 0.

A picture



The (sub)probabilistic powerdomain $\mathcal{V}_{\leq 1}(P)$

A valuation is a map $\mu : \mathcal{O}(P) \to \overline{\mathbb{R}}_+$ such that:

$$\begin{array}{lcl} \mu(\emptyset) & = & 0 \\ \mu(U \cup V) + \mu(U \cap V) & = & \mu(U) + \mu(V) \end{array}$$

It is subprobabilistic if $\mu(P) \leq 1$.

Example: the Dirac valuation $\delta_P(x)$ ($x \in P$), where

$$\delta_x(V) = \chi_V(x)$$

The (sub)probabilistic powerdomain $\mathcal{V}_{\leq 1}(P)$ consists of all Scott-continuous such valuations, with the pointwise order.

It is a Kegelspitze with the pointwise probabilistic choice operations, action, and zero.

Example

$$\mathcal{V}_{\leq 1}(\mathbb{1}) \cong \mathbb{I}$$

Integration

For any Scott-continuous valuation μ and Scott-continuous function $f : P \to \overline{\mathbb{R}}_+$ there is an integral

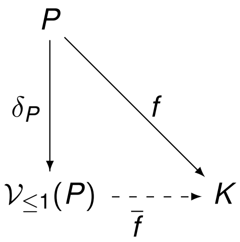
$$\int f \, d\mu \;\in\; \overline{\mathbb{R}}_+$$

It can be defined by a Choquet integral:

$$\int f \, d\mu = \int_0^{+\infty} \mu(f^{-1}(]r, +\infty])) \, dr$$

More on $\mathcal{V}_{\leq 1}(P)$

- $\mathcal{V}_{\leq 1}(P)$ is a domain if P is, and then
- $\mathcal{V}_{\leq 1}(P)$ is the free Kegelspitze over P:



Distributing probabilistic choice over nondeterministic choice

Distributive law

$$x +_r (y \cup z) = (x +_r y) \cup (x +_r z)$$

Convexity

$$x \cup (x +_r y) \cup y = x \cup y$$

This follows from the distributive law:

$$\begin{aligned}
x \cup y &= (x \cup y) +_r (x \cup y) \\
&= (x +_r x) \cup (x +_r y) \cup (y +_r x) \cup (y +_r y) \\
&= x \cup (x +_r y) \cup (y +_r x) \cup y \\
&\supseteq x \cup (x +_r y) \cup y \\
&\supseteq x \cup y
\end{aligned}$$

Semilattice Kegelspitzen

- A Kegelspitze semilattice is a Kegelspitze equipped with a semilattice operation $\cup$ over which convex combinations distribute.

- It is a Kegelspitze join-semilattice if $\cup$ is the binary supremum operation (equivalently, if $x \leq x \cup y$ always holds).

- Example $\mathbb{I}$ is a Kegelspitze join-semilattice.

Convex subsets of a Kegelspitze K

- $X \subseteq K$ is convex if it is closed under the barycentric operations, i.e., for all $x, y \in X$, $r \in [0, 1]$ we have $x +_r y \in X$.
- If X_i is a directed collection of convex sets then their union

$$\bigcup_i X_i$$

  is also convex.
- If $X, Y \subseteq K$ are convex, so is

$$X +_r Y = \{x +_r y \mid x \in X, y \in Y\}$$

  as

$$(x +_r y) +_s (z +_r w) = (x +_s z) +_r (y +_s w)$$

- If $X \subseteq K$ is convex, so is its Scott closure $\overline{X}$.
- We write $\mathrm{conv}(X)$ for the least convex set containing $X \subseteq K$

The Hoare power Kegelspitze $\mathcal{H}(K)$

- Elements Non-empty convex, Scott-closed subsets of K.
- Order structure This is subset and for any directed set X_i:

$$\bigvee_{i\in I}^{\uparrow} X_i = \overline{\bigcup_{i\in I}^{\uparrow} X_i}$$

- Barycentric structure

$$X +_{r\mathcal{H}(K)} Y = \overline{X +_r Y}$$

- Zero

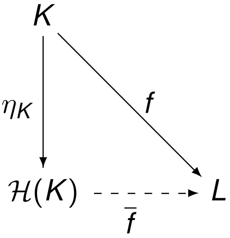
$$0_{\mathcal{H}(K)} = \{0_K\}$$

- Semilattice structure

$$X \cup_{\mathcal{H}(K)} Y = \overline{\mathrm{conv}(X \cup Y)}$$

Properties of $\mathcal{H}(K)$

- $\mathcal{H}(K)$ is the free Kegelspitze join-semilattice over K:



where

$$\eta_K(x) = \downarrow x \text{ and } \bar{f}(X) = \bigvee_{x \in X} f(x)$$

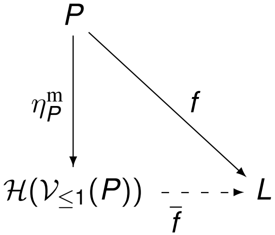
- $\mathcal{H}(K)$ is a domain if K is.

The Hoare subprobabilistic powerdomain

This is

$$\mathcal{H}_{\mathrm{m}}(P) = \mathcal{H}(\mathcal{V}_{\leq 1}(P))$$

- If P is a domain, $\mathcal{H}_{\mathrm{m}}(P)$ is the free Kegelspitze join-semilattice over P:



where $\eta_P^{\mathrm{m}} = \eta_{\mathcal{V}_{\leq 1}} \circ \delta_P$.

- $\mathcal{H}_{\mathrm{m}}(P)$ is a domain if P is.

More context

- Our inequational classes of algebras are closed under Scott-continuous function spaces A^P, equipped with the pointwise structure.
- If the axioms are commutative (aka entropic) then the the homomorphisms $[A, B]$ between two algebras in the class are also in it (again using the pointwise structure),
- and the free algebra extension map is an algebra isomorphism.
- The semilattice axioms are commutative:

$$\vdash (x_{11} \cup x_{12}) \cup (x_{21} \cup x_{22}) = (x_{11} \cup x_{21}) \cup (x_{21} \cup x_{22})$$

- The Kegelspitze axioms are commutative:

$$\vdash (x_{11} +_r x_{12}) +_s (x_{21} +_r x_{22}) = (x_{11} +_s x_{21}) +_r (x_{21} +_s x_{22})$$

$$\vdash 0 +_r 0 = 0$$

- The axioms for semilattice Kegelspitzen are not:

$$\not\vdash (x_{11} +_r x_{12}) \cup (x_{21} +_r x_{22}) = (x_{11} \cup x_{21}) +_r (x_{21} \cup x_{22})$$

Predicate transformers and functional representations

Want a bijection or isomorphism with state transformers:

$$\frac{P \longrightarrow \mathcal{H}(Q)}{\mathcal{O}(Q) \xrightarrow{\text{laws}} \mathcal{O}(P)} \qquad \text{equivalently} \qquad \frac{P \longrightarrow \mathcal{H}(Q)}{\mathbb{S}^Q \xrightarrow{\text{laws}} \mathbb{S}^P}$$

Setting $P = \mathbb{1}$ we get a functional representation of $\mathcal{H}$:

$$\Lambda : \mathcal{H}(Q) \;\cong\; \mathbb{S}^Q \xrightarrow{\text{laws}} \mathbb{S}$$

Conversely we have

$$\frac{\dfrac{P \longrightarrow \mathcal{H}(Q)}{P \longrightarrow (\mathbb{S}^Q \xrightarrow{\text{laws}} \mathbb{S})}}{\mathbb{S}^Q \xrightarrow{\text{laws}} \mathbb{S}^P}$$

Functional representation of the Hoare powerdomain

Let P be a dcpo.

The functional representation of $\mathcal{H}(P)$ associates to any closed subset the collection of open sets intersecting it.

Specifically $\mathbb{S}$ — and so $\mathbb{S}^P$ — is a Scott-continuous semilattice and we have a semilattice isomorphism:

$$\Lambda : \mathcal{H}(P) \cong [\mathbb{S}^P, \mathbb{S}]$$

where

$$\Lambda(C)(f) = \bigvee_{x \in C} f(x)$$

or

$$\Lambda(C)(V) = \begin{cases} \top & (C \cap V \neq \emptyset) \\ \bot & (C \cap V = \emptyset) \end{cases}$$

Note, up to iso:

$$\mathbb{S} = \mathcal{H}(\mathbb{1}) \qquad \Lambda(C)(f) = \overline{f}(C)$$

Hoare powerdomain predicate transformers

Let P, Q be dcpos.

We have semilattice isomorphisms:

$$\frac{\dfrac{P \longrightarrow \mathcal{H}(Q)}{P \longrightarrow [\mathbb{S}^Q, \mathbb{S}]}}{\mathbb{S}^Q \xrightarrow{\text{hom}} \mathbb{S}^P}$$

Functional representation of the subprobabilistic powerdomain

Let P be a dcpo.

Integration provides a Scott-continuous bilinear function

$$\int : \mathbb{I}^P \times \mathcal{V}_{\leq 1}(P) \to \mathbb{I}$$

which gives a Kegelspitze isomorphism

$$\Lambda : \mathcal{V}_{\leq 1}(P) \;\cong\; [\mathbb{I}^P, \mathbb{I}]$$

where

$$\Lambda(\mu)(f) = \int f \, d\mu$$

It has inverse

$$\Lambda^{-1}(F) = V \mapsto F(\chi_V)$$

Note, as before, up to iso:

$$\mathbb{I} = \mathcal{V}_{\leq 1}(\mathbb{1}) \qquad \Lambda(\mu)(f) = \overline{f}(\mu)$$

Subprobability powerdomain predicate transformers

Let P, Q be dcpos.

We have Kegelspitze isomorphisms:

$$\frac{\dfrac{P \longrightarrow \mathcal{V}_{\leq 1}(Q)}{P \longrightarrow [\mathbb{I}^Q, \mathbb{I}]}}{\mathbb{I}^Q \xrightarrow{\text{linear}} \mathbb{I}^P}$$

Sublinear maps

- A Kegelspitze map $f : K \to K'$ is
  - homogeneous if
    $$f(r \cdot x) = r \cdot f(x) \quad (x \in K, r \in \mathbb{I})$$
  - convex if
    $$f(x +_r y) \leq f(x) +_r f(y) \quad (x, y \in K, r \in \mathbb{I})$$
  - sublinear if it is both homogeneous and convex.

- Let K be a Kegelpspitze and L be a join-semilattice Kegelspitze. The sublinear maps $K \longrightarrow L$ are closed under arbitrary sups. The linear maps are not.

- The Scott-continuous sublinear such maps form a join-semilattice Kegelspitze $[K, L]_{\mathrm{s}}$.

Functional representation of the Hoare mixed powerdomain

We aim to prove:

Theorem

Let P be a domain. Then we have a Kegelspitze join-semilattice isomorphism

$$\Lambda_P \colon \mathcal{H}(\mathcal{V}_{\leq 1}(P)) \;\cong\; [\mathbb{I}^P, \mathbb{I}]_{\mathrm{s}}$$

where

$$\Lambda_P(X)(f) = \sup_{\mu \in X} \int f \, d\mu$$

Remark $\Lambda_P(X)$ is the sup of the linear evaluation maps $\mathrm{ev}(x)$ ($x \in X$), where

$$\mathrm{ev}(x)(f) = f(x)$$

Hoare mixed powerdomain predicate transformers

Let P be a dcpo and Q be a domain.

We have Kegelspitze join-semilattice isomorphisms:

$$\dfrac{\dfrac{P \longrightarrow \mathcal{H}(\mathcal{V}_{\leq 1}(Q))}{P \longrightarrow [\mathbb{I}^Q, \mathbb{I}]_{\mathrm{s}}}}{\mathbb{I}^P \xrightarrow{\text{sublinear}} \mathbb{I}^P}$$

Proof strategy to obtain functional representation

- I: Introduce d-cones.
- II: Define Hoare powercones and quote result for them from [KPT].
- III: Develop relation between Kegelspitzen and d-cones.
- IV: Infer result for Hoare power Kegelspitzen.
- V: Specialise to mixed power domains.

I: d-cones

A d-cone C is a commutative monoid with an $\overline{\mathbb{R}}_+$-action, continuously parameterised in $\overline{\mathbb{R}}_+$:

Commutative Monoid

$$\begin{cases} (x+y)+z & = & x+(y+z) \\ x+y & = & y+x \\ x+0 & = & 0 \end{cases}$$

Action

$$\begin{cases} (r+s)\cdot x & = & r\cdot x + s\cdot x \\ 0\cdot x & = & 0 \\ & & \\ rs\cdot x & = & r\cdot(s\cdot x) \\ 1\cdot x & = & x \end{cases}$$

All d-cones are Kegelspitzen, and a map $C \to D$ is linear (i.e., a d-cone morphism) iff it is a Kegelspitze homomorphism.

I: The valuation powerdomain

The valuation powerdomain $\mathcal{V}(P)$ consists of all Scott-continuous valuations, with the pointwise order.

$\mathcal{V}(P)$ is a d-cone with the pointwise operations:

$$(\mu+\nu)(V) = \mu(V)+\nu(V) \qquad 0(V) = 0 \qquad (r\cdot\mu)(V) = r\cdot(\mu(V))$$

$\mathcal{V}(P)$ is a domain if P is.

Example $\overline{\mathbb{R}}_+$ and then (pointwise) the Scott-continuous function space $\mathcal{L}(P) =_{\mathrm{def}} \overline{\mathbb{R}}_+^P$.

I: Functional representation of the valuation powerdomain

Let P be a dcpo.

As before, integration provides a Scott-continuous bilinear function

$$\mathbb{I} : \mathcal{L}(P) \times \mathcal{V}(P) \to \overline{\mathbb{R}}_+$$

which gives a d-cone isomorphism

$$\Lambda : \mathcal{V}(P) \cong [\mathcal{L}(P), \overline{\mathbb{R}}_+]$$

where

$$\Lambda(\mu)(f) = \int f \, d\mu$$

but we will need more ......

I: Duality and the valuation powercone

- The dual of a d-cone is $C^* =_{\mathrm{def}} [C, \overline{\mathbb{R}}_+]$
- We already have
$$\mathcal{V}(P) \cong \mathcal{L}(P)^*$$
- If P is continuous then (currying) integration also gives us
$$\mathcal{L}(P) \cong \mathcal{V}(P)^*$$
- So, if P is continuous then $\mathcal{V}(P)$ is reflexive where
- A d-cone C is reflexive if
$$\mathrm{ev} : C \cong C^{**}$$
where
$$\mathrm{ev}(c) = f \mapsto f(x)$$
is the evaluation functional.

II: Lower semilattice d-cones and Hoare powercones

- C is a join-semilattice d-cone if it has binary sups such that

$$x + (y \vee z) = (x + y) \vee (x + z) \quad (x, y, z \in C)$$

$$r \cdot (x \vee y) = r \cdot x \vee r \cdot y \quad (r \in \overline{\mathbb{R}}_+, x, y \in C)$$

  $\overline{\mathbb{R}}_+$ provides an example.

- The Hoare powercone $\mathcal{H}(C)$ consists of the non-empty convex, Scott-closed subsets of C ordered by $\subseteq$.
  - Directed sups are given by

$$\bigvee_{i \in I}^{\uparrow} X_i = \overline{\bigcup_{i \in I}^{\uparrow} X_i}$$

  - The d-cone structure is given by;

$$X + Y =_{\mathrm{def}} \overline{X + Y} \qquad 0 = \{0\}$$

  - binary sups are given by

$$X \vee Y = \overline{\mathrm{conv}(X \cup Y)}$$

- $\mathcal{H}(C)$ is the free join-semilattice d-cone over C and continuous if C is.

II: Sublinear maps again

- A map $f : C \to D$ is
  - homogeneous if
    $$f(r \cdot x) = r \cdot f(x) \quad (x \in C, r \in \overline{\mathbb{R}}_+)$$
  - subadditive if
    $$f(x + y) \leq f(x) + f(y) \quad (x, y \in C)$$
  - sublinear if it is both homogeneous and subadditive
- Agrees with Kegelspitze definitions.
- If C is a d-cone and L is a semilattice d-cone then the Scott-continuous sublinear maps from C to L form a d-cone join-semilattice $[C, L]_{\mathrm{s}}$.

II: Functional representation of the Hoare powercone

Theorem

Let C be a reflexive continuous d-cone with a continuous dual. Then we have a d-cone join-semilattice isomorphism

$$\Lambda_C \colon \mathcal{H}(C) \cong [C, \overline{\mathbb{R}}_+]_{\mathrm{s}}$$

where

$$\Lambda_C(X) = \sup_{x \in X} f(x)$$

III: Embedding Kegelspitzen in d-cones

- A Scott-continuous map $f : P \to Q$ is an order-embedding if

$$f(x) \leq f(y) \implies x \leq y \quad (x, y \in P)$$

- A Kegelspitze K satisfies condition (OC3) if

$$x \leq r \cdot y \implies \exists x' \in K.\, x = r \cdot x' \quad (r \in \mathbb{I}, x, y \in K)$$

  Examples Any d-cone; $\mathcal{V}_{\leq 1}(P)$.
- Such a K has a universal order-embedding in a d-cone

$$K \xhookrightarrow{e} \mathrm{d\text{-}Cone}(K)$$

  as a Scott-closed subset. The extension map

$$f \in [K, C] \mapsto [\mathrm{d\text{-}Cone}(K), C]$$

  is an isomorphism of d-cones, with the pointwise d-cone structure on $[K, C]$ (!).
- And $\mathrm{d\text{-}Cone}(K)$ is continuous if K is.
- Example If P is a domain:

$$\mathcal{V}_{\leq 1}(P) \hookrightarrow \mathcal{V}(P)$$

IV: Norms and nonexpansive maps

- A norm on a d-cone C is a sublinear Scott-continuous map $\| \; \| : C \to \overline{\mathbb{R}}_+$.

- Examples The identity on $\overline{\mathbb{R}}_+$ and $f \mapsto \bigvee_{x \in P} f(x)$ on $\mathcal{L}(P)$.

- A function $F : C \to D$ between normed d-cones is nonexpansive if

$$\|F(x)\| \leq \|x\| \quad (x \in C)$$

- Suppose C and D are normed d-cones and D is also a d-cone join-semilattice. Then the Scott-continuous nonexpansive sublinear maps from C to D form a Kegelspitze $[K, L]_{\mathrm{ns}}$.

IV: Functional representation of Hoare power Kegelpspitzen

The dual of a Kegelspitze K is the d-cone $K^* =_{\text{def}} [K, \overline{\mathbb{R}}_+]$. It has a norm given by:

$$\|f\| = \bigvee_{x \in K} f(x)$$

Theorem

Let K be a continuous Kegelspitze satisfying (OC3) *such that K^* is continuous and* $\text{d-Cone}(K)$ *is reflexive.*

Then we have a Kegelspitze join-semilattice isomorphism

$$\Lambda_K \colon \mathcal{H}(K) \cong [K^*, \overline{\mathbb{R}}_+]_{\text{ns}}$$

where

$$\Lambda_K(X)(f) = \sup_{x \in X} f(x)$$

IV: Beginning the proof

We assume we have a universal order embedding $K \overset{e}{\hookrightarrow} C$ of K as a Scott-closed subset of a continuous d-cone C.

Define a norm on C^* by:

$$\|f\| = \bigvee_{x \in K} f(k)$$

Recalling $\mathrm{ev} : C \to C^{**}$ where $\mathrm{ev}(x)(f) = f(x)$ we have:

Lemma

- $\mathrm{ev}(x)$ *is nonexpansive iff* $x \in K$*, and so*
- *if* C *is reflexive then the nonexpansive* $F \in C^{**}$ *are the* $\mathrm{ev}(x)$ *with* $x \in K$*.*

For example, if $x \in K$ then:

$$\|\mathrm{ev}(x)(f)\| = \mathrm{ev}(x)(f) = f(x) \leq \bigvee_{x \in K} f(x) = \|f\|$$

IV: More of the proof

$$
\begin{array}{ccc}
\mathcal{H}(K) & \xrightarrow{\Lambda_C \restriction \mathcal{H}(K)} & [C^*, \overline{\mathbb{R}}_+]_{\mathrm{ns}} \\
\cap\big\downarrow \mathcal{H}(e) & & \cap\big\downarrow \\
\mathcal{H}(C) & \xrightarrow[\Lambda_C]{} & [C^*, \overline{\mathbb{R}}_+]_{\mathrm{s}}
\end{array}
$$

$\Longrightarrow$ Assume $X \subseteq K$.

Then:

$$\|\Lambda_C(X)(f)\| = \Lambda_C(X)(f) = \bigvee_{x \in X} f(x) \leq \bigvee_{x \in K} f(x) = \|f\|$$

$\Longleftarrow$ Assume $\Lambda_C(X)$ nonexpansive.

Then, if $x \in X$, we have $\mathrm{ev}(x) \leq \Lambda_C(X)$. So $\mathrm{ev}(x)$ is nonexpansive. So $x \in K$. So $X \subseteq K$.

IV: Rest of proof

The universal inclusion

$$K \overset{e}{\hookrightarrow} C$$

yields an isomorphism of normed d-cones

$$C^* \cong K^*$$

which gives an isomorphism of join-semilattice Kegelspitzen

$$[C^*, \overline{\mathbb{R}}_+]_{\mathrm{ns}} \cong [K^*, \overline{\mathbb{R}}_+]_{\mathrm{ns}}$$

which, with previous, gives the isomorphism

$$\mathcal{H}(K) \xrightarrow{\Lambda_C \restriction \mathcal{H}(K)} [C^*, \overline{\mathbb{R}}_+]_{\mathrm{ns}} \cong [K^*, \overline{\mathbb{R}}_+]_{\mathrm{ns}}$$

which is

$$\Lambda_K : \mathcal{H}(K) \cong [K^*, \overline{\mathbb{R}}_+]_{\mathrm{ns}}$$

the required isomorphism of join-semilattice Kegelspitzen.

V: Representation of Hoare mixed powerdomain

Let P be a domain.

- Then $\mathcal{V}_{\leq 1}(P)$ is continuous; $\text{d-Cone}(\mathcal{V}_{\leq 1}(P)) \cong \mathcal{V}(P)$ is reflexive; and the dual of $\mathcal{V}_{\leq 1}(P)$ is $\mathcal{V}_{\leq 1}(P)^* \cong \mathcal{V}(P)^* \cong \mathcal{L}(P)$ and so is continuous.
- So we have an isomorphism of join-semilattice Kegelspitzen

$$\Lambda_{\mathcal{V}_{\leq 1}(P)} : \mathcal{H}(\mathcal{V}_{\leq 1}(P)) \cong [\mathcal{V}_{\leq 1}(P)^*, \overline{\mathbb{R}}_+]_{\text{ns}}$$

- Have isomorphisms of normed d-cones:

$$\mathcal{V}_{\leq 1}(P)^* \cong \mathcal{V}(P)^* \cong \mathcal{L}(P)$$

- which gives the desired isomorphism of join-semilattice Kegelspitzen

$$\Lambda_P : \mathcal{H}(\mathcal{V}_{\leq 1}(P)) \cong [\mathcal{L}(P), \overline{\mathbb{R}}_+]_{\text{ns}} \cong [\mathbb{I}^P, \mathbb{I}]_{\text{s}}$$

Why can't we proceed directly?

- Somehow replace the cone by a Kegelspitze and $\overline{\mathbb{R}}_+$ by $\mathbb{I}$ in the following?

Theorem

(Separation Theorem [TKP]) Let D be a continuous d-cone with two disjoint nonempty convex subsets C and V, with C Scott-closed and V Scott-open.

Then there exists a Scott-continuous linear functional $f : C \to \overline{\mathbb{R}}_+$ such that

$$f(x) \leq 1 < f(y)$$

for all $x \in C$ and all $y \in V$.

- In other cases (upper, convex) don't immediately know how to formulate in $\mathbb{I}$ terms.
- In the various cases it is already complex in d-cone case, so at least current 'piggyback' approach is efficient.

The other distributivity law

This is

$$(D') \qquad x \cup (y +_r z) = (x \cup y) +_r (x \cup z) \qquad (r \in \mathbb{I})$$

Let S, B be the theories of semilattices, barycentric algebras, respectively.

Theorem

The theory $\mathrm{S} + \mathrm{B} + \mathrm{D}$ is equivalent to the theory of two semilattices, with one distributing over the other.

Lemma

Suppose T extends B. Then:

$$\frac{\vdash_T \; t +_r u = t +_s u}{\vdash_T \; t +_p u = t +_q u}$$

where $0 < r < s < 1$ *and* $0 < p < q < 1$

Proof (all reals in $]0, 1[$)

Substituting $(y +_r z)$ for x in D' (and using S and re-using D') we get (E1)

$$(y+_r z) = ((y+_r z)\cup y)+_r((y+_r z)\cup z) = (y+_r(y\cup z))+_r((y\cup z)+_r z)$$

Substituting $y \cup z$ for z (and using S and then B) we get:

$$y +_r (y \cup z) = (y +_r (y \cup z)) +_r ((y \cup z) +_r (y \cup z)) = y +_{r^2} (y \cup z)$$

So by the Lemma we get (E2):

$$y +_r (y \cup z) = y +_s (y \cup z)$$

But then:

$$\begin{array}{rcll} (y +_r z) & = & (y +_r (y \cup z)) +_r ((y \cup z) +_r z) & \text{(by (E1))} \\ & = & (y +_{r'} (y \cup z)) +_r ((y \cup z) +_{r'} z) & \text{(by (E2), and B)} \\ & = & (y +_r (y \cup z)) +_{r'} ((y \cup z) +_r z) & \text{(by B)} \\ & = & (y +_{r'} (y \cup z)) +_{r'} ((y \cup z) +_{r'} z) & \text{(by (E2), and B)} \\ & = & (y +_{r'} z) & \text{(by (E1))} \end{array}$$

So

$$y +_r z = y +_s z$$