Some cardinal functions on Boolean algebras
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Introduction

A Boolean algebra (BA) can be defined as any structure $(A, +, \cdot, -, 0, 1)$ isomorphic to a structure $(B, \cup, \cap, \backslash, \emptyset, X)$ where B is a collection of subsets of X closed under these operations. We describe some functions defined on the class of all infinite Boolean algebras and taking cardinal numbers as values. The functions considered, and many others, are studied thoroughly in Monk, **Cardinal Invariants on Boolean Algebras**, 2nd Revised edition, Birkhäuser 2014; we refer to this book by [M]. Here we indicate some of the important problems, and we describe additional results obtained since the publication of that book.

We assume familiarity with Boolean algebras. A good background is the book of Sabine Koppelberg, **General theory of Boolean algebras,** North-Holland 1989. Unless otherwise indicated, all Boolean algebras mentioned are assumed to be infinite.

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Free products

If A and B are BAs, their *free product* $A \oplus B$ is an algebra C generated by isomorphic copies A' and B' of A and B such that $a \cdot b \neq 0$ if $a, b \neq 0$, with $a \in A'$ and $b \in B'$. Free products always exist. They are actually coproducts in the category of Boolean algebras with homomorphisms.

For some cardinal functions k the equality $k(A \oplus B) = \min(k(A), k(B))$ holds; this is true for k the tower number for example. The tower number $tow(A)$ is the least size of a strictly increasing sequence of elements of A with supremum 1. If $\langle a_\alpha : \alpha < \kappa \rangle$ is such a sequence in A, it is also such in $A \oplus B$, giving by symmetry the inequality $\leq$. Equality is more difficult. The equality is open for most other functions. For example, let $\mathfrak{a}(A)$ be the least size of an infinite partition of A; that is, the least size of an infinite set of pairwise disjoint elements of A with supremum 1.

Problem 1. *Is* $\mathfrak{a}(A \oplus B) = \min(\mathfrak{a}(A), \mathfrak{a}(B))$*?*

Ideal independence

A subset X of a BA A is *ideal independent* if for any $x \in X$ and $a_0, \ldots, a_{m-1} \in X \backslash \{x\}$ we have $x \not\leq a_0 + \cdots + a_{m-1}$. The smallest ideal containing $X \backslash \{x\}$ is the set of all subelements of sums of elements of $X \backslash \{x\}$. This explains the name.

We define $\mathrm{s_{mm}}(A)$ to be the least size of a maximal ideal independent subset of A. We use the letter s because of the connection with spread, a topological notion. In fact, ideal independence corresponds by Boolean duality to discrete subsets.

For any BAs A, B, the product $A \times B$ is the set of all pairs (a, b) with $a \in A$, $b \in B$, and the natural operations; e.g. $(a, b) + (c, d) = (a + c, b + d)$. It is easy to see that $\mathrm{s_{mm}}(A \times B) \leq \min(\mathrm{s_{mm}}(A), \mathrm{s_{mm}}(A))$.

Problem 2 *Is* $\mathrm{s_{mm}}(A \times B) = \min(\mathrm{s_{mm}}(A), \mathrm{s_{mm}}(A))$*.?*

For free products, no relationship between $\mathrm{s_{mm}}(A \oplus B)$, $\mathrm{s_{mm}}(A)$, and $\mathrm{s_{mm}}(A)$ is known.

Ideal independence 2

In [M] it is shown that $\mathfrak{r}(A) \le \mathrm{s_{mm}}(A)$ and $\mathfrak{p}(A) \le \mathrm{s_{mm}}(A)$ (due to Bruns). Here for any BA B, $\mathfrak{r}$ is the reaping number—the smallest size of a subset X of B such that for every nonzero $a \in B$ there is an $x \in X$ such that $0 \neq x \le a$ or $0 \neq x \le -a$. Also $\mathfrak{p}(B)$ is the smallest size of a subset X of B such that $\sum X = 1$, while $\sum F \neq 1$ for every finite subset of B. Selker [15] has shown under CH that there is an atomless BA A such that $\mathrm{s_{mm}}(A) < \mathfrak{u}(A)$, where $\mathfrak{u}(A)$ is the least size of a set filter-generating a nonprincipal ultrafilter of A.

Ideal independence 3

For the rest of this section we consider the special case when $A = \mathscr{P}(\omega)/\text{fin}$, and omit mention of A. For this algebra, $\mathrm{s_{mm}}$ is a new function, apparently different from the many usual function studied for this algebra. Easy arguments show that $\omega_1 \leq \mathrm{s_{mm}} \leq 2^\omega$, and under Martin's axiom $\mathrm{s_{mm}} = 2^\omega$. The above inequality $\mathfrak{p}(B) \leq \mathrm{s_{mm}}(B)$ for general BAs was generalized by Cancino, Guzmán, and Miller (unpublished) as far as A is concerned to $\mathfrak{d} \leq \mathrm{s_{mm}}$, where $\mathfrak{d}$ is the smallest size of a subset X of ${}^\omega\omega$ such that for every $f \in {}^\omega\omega$ there is a $g \in X$ such that $f(m) < g(m)$ for all $m \in \omega$. It is known that $\mathfrak{p} \leq \mathfrak{d}$.
A forcing argument shows that it is relatively consistent to have $\mathrm{s_{mm}} < 2^\omega$. This forcing argument uses the following general lemma.

Ideal independence, 4

Lemma. *Let M be a countable transitive model of ZFC, and suppose that I is an ideal in $\mathscr{P}(\omega)^M$ containing all singletons. Define $P = \{(b, y) : b \in I,\ y \in [\omega]^{<\omega}\}$; $(b, y) \le (b', y')$ if $b \supseteq b'$, $y \supseteq y'$, $y \cap b' \subseteq y'$. Then P is ccc. Let G be P-generic over M, and define $d = \bigcup_{(b,y)\in G} y$. Then the following conditions hold:*

(i) If $c \subseteq \omega$, $c \in M$, and $c \notin I$, then $c \cap d$ is infinite.
(ii) If $c \subseteq \omega$, $c \in M$, and $c \notin I$ then $c \backslash d$ is infinite.
(iii) If $b \in I$, then $b \cap d$ is finite.

Now the consistency result $\mathrm{s_{mm}} < 2^\omega$ is obtained by iterated forcing, the successor step working with a system $\langle [a_\alpha] : \alpha < \kappa \rangle$ of ideal independent elements of $\mathscr{P}(\omega)/\mathrm{fin}$, defining I to be the ideal generated by the singletons together with all elements $a_\alpha \cap a_\beta$ for $\alpha \neq \beta$.

Ideal independence, 5

Cancino, Guzmán, and Miller also proved the relative consistency of $\mathrm{s_{mm}} = \mathfrak{u} = \mathfrak{d} = \omega_1 < 2^\omega$, the relative consistency of $\mathrm{s_{mm}} < \mathfrak{i}$, and several other relative consistency results. Here $\mathfrak{i}(B)$ is the least size of a maximal independent set, where X is independent if for any finite disjoint subsets F, G of X we have $\prod_{a\in F} \cdot \prod_{b\in G} -b \neq 0$. This consistency result $\mathrm{s_{mm}} < \mathfrak{i}$ solves, consistently, Problem 126 in [M]. However, the following questions are open.

Problem 3 *Is* $\mathfrak{u}(\mathscr{P}(\omega)/\mathrm{fin}) \leq \mathrm{s_{mm}}(\mathscr{P}(\omega)/\mathrm{fin})$*?*

Problem 4 *Is* $\mathrm{s_{mm}}(A) \leq \mathfrak{i}(A)$ *for any atomless BA A?*

Maximal free sequences

A sequence $\langle a_\xi : \xi < \alpha \rangle$ of elements of a BA A is free if for any finite $F, G \subseteq \alpha$ such that $\forall \xi \in F \forall \eta \in G [a_\xi < a_\eta]$ we have $\prod_{\xi \in F} a_\xi \cdot \prod_{\eta \in G} -a_\eta \neq 0$. The sequence is said to be maximal if it cannot be extended to another free sequence $\langle a_\xi : \xi < \beta \rangle$ with $\alpha < \beta$. We define $\mathfrak{f}(A)$ to be the minimum of $|\alpha|$ for which there is a maximal free sequence of length α. As with $\mathrm{s_{mm}}$ it is easy to see that $\mathfrak{r}(A), \mathfrak{p}(A) \leq \mathfrak{f}(A)$. Selker [15] has constructed under CH an atomless BA A such that $\mathfrak{f}(A) < \mathfrak{u}(A)$.

Problem 5 *Is there an atomless BA A such that* $\mathfrak{u}(A) < \mathfrak{f}(A)$*?*

Again we now consider the special algebra $A = \mathscr{P}(\omega)/\mathrm{fin}$, omitting "$A$". Simple results about $\mathfrak{f}$ together with a construction of Kunen show that it is relatively consistent to have $\mathfrak{f} < 2^\omega$; in his model we actually have $\mathfrak{f} = \mathfrak{u}$.

Problem 6 *Is it consistent to have* $\mathfrak{f} \neq \mathfrak{u}$*?*

Irredundance

A subset X of a BA A is said to be irredundant if for each $x \in X$, x is not in the subalgebra of A generated by $X \setminus \{x\}$. We let $\mathrm{irr}_{\mathrm{mm}}(A)$ be the least size of a maximal irredundant subset of A. It is easy to see that if A is a subalgebra of $\mathscr{P}(\kappa)$, κ infinite, and A contains each set α for $\alpha < \kappa$, then $\mathrm{irr}_{\mathrm{mm}}(A) = \kappa$. In [M] an atomless BA A is constructed such that $\mathrm{irr}_{\mathrm{mm}}(A) = \omega$ and $|A| = 2^{\omega}$. Kunen (unpublished) showed that, assuming $2^{\aleph_1} = \aleph_2$ there is an atomic BA $A \subseteq \mathscr{P}(\omega_1)$ containing all singletons such that $\mathrm{irr}_{\mathrm{mm}}(A) > \aleph_1$. This answers negatively Problem 90 of [M]. A subset X of a BA A is dense if for every nonzero $a \in A$ there is a nonzero $x \in X$ such that $x \leq a$. The density of A is the smallest size $\pi(A)$ of a dense subset of A. It is easy to see that $\pi(A) \leq \mathrm{irr}_{\mathrm{mm}}(A)$. The following is a revised version of Problem 90 of [M].

Problem 7 *Is* $\pi(A) = \mathrm{irr}_{\mathrm{mm}}(A)$ *for every atomless BA A?*

Incomparability

We define $\mathrm{inc_{mm}}(A)$ to be the least size of a maximal set of pairwise incomparable elements of A. In unpublished work, Charles Scherer has shown that if A is atomless and has a countable dense subset, then $\mathrm{inc_{mm}}(A) = \omega$.

Now again we consider the special case $A = \mathscr{P}(\omega)/\mathrm{fin}$ and we omit A. Hrusak (unpublished) has shown that $\mathrm{inc_{mm}} = 2^\omega$. This answers negatively Problem 157 of [M]. We give the proof here. It depends on the following general fact about BAs. For any BA B and any $a \in B$, let $B \restriction a = \{x \in B : x \leq a\}$. This is a BA under the operations $+$, $\cdot$ of B, with $-^{B\restriction a}x = a \cdot -x$ and with unit element a. Since $\pi(A \restriction a) = 2^\omega$ for each nonzero $a \in A$, the following theorem gives $\mathrm{inc_{mm}} = 2^\omega$.

Incomparability, 2

Theorem. *If B is an infinite BA, $X \subseteq B$, X incomparable, $0, 1 \notin X$, and $|X| < \min\{\pi(B \upharpoonright a) : a \in B, a \neq 0\}$, then there is a $b \in B$ which is incomparable to each element of X.*

Proof. Assume the hypotheses. Let C be the subalgebra of B generated by X; then also $|C| < \min\{\pi(B \upharpoonright a) : a \in B, a \neq 0\}$. Take any $a \in X$. Then $a \neq 0, 1$. Since $|C| < \pi(B \upharpoonright -a)$, there is an $x \leq -a$ with $x \neq 0$ such that no nonzero element of C is below x. Similarly there is a $y \leq a$ with $y \neq 0$ such that no nonzero element of C is below y. Let $b = x + a \cdot -y$. Note that $a \not\leq b$. For, suppose that $a \leq b$. Also $a \leq -x$, so $a \leq b \cdot -x \leq -y$, hence $y \leq -a$ and so $y = 0$, contradiction. Also, $b \not\leq a$. For if $b \leq a$ then $x \leq a$, so $x = 0$, contradiction.

Incomparability, 3

Suppose that $c \in X$ and $c \leq b$. So $c \neq a$, hence $c \cdot -a \neq 0$. Then $c \cdot -a \leq x$, $c \cdot -a \in C$, and $c \cdot -a \neq 0$, contradiction. Suppose that $c \in X$ and $b \leq c$. Then $a \neq c$, so $a \cdot -c \neq 0$. Now $a \cdot -y \leq c$, so $a \cdot -c \leq y$, again a contradiction. So b is incomparable with each element of X.

References

Selker [15] *Ideal independence, free sequences, and the ultrafilter number.* Comment. Math. Univ. Carol. 56, no. 1 (2015), 117–124.