A contextual generalization of Boolean Algebras

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Motivation

- **Formal Concept Analysis (FCA)** started in the eighties as applied lattice theory (by Rudolf Wille) and rapidly established himself with many applications in conceptual clustering, rule generation, rule and web mining, lattice theory,. . .
- FCA is based on the the formalization of the notion of *concept*.
- Traditional philosophers considered a **concept** to be determined by its extent and its intent. The **extent** consists of all objects belonging to the concept while the **intent** is the set of all attributes shared by all objects of the concept.
- Concepts are considered as basic unit of thought. So our aim is to develop a logic based on concepts as units of thought.

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- Formalize concepts and logical operations on concepts
- Investigate the related structures

For this talk:

- Formal Concept Analysis
- The problem of negation
- Weak negation and related structures:
  - Double Boolean algebras
  - Weakly dicomplemented lattices

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Universe

- A **formal context** is a triple $\mathbb{K} := (G, M, \mathrm{I})$ such that $\mathrm{I} \subseteq G \times M$.
- $G :\equiv$ set of **objects** $\qquad M :\equiv$ set of **attributes**.
- $g \,\mathrm{I}\, m :\iff (g, m) \in \mathrm{I}$. $\qquad g$ has attribute m.

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concepts

- $G \supseteq A \mapsto A' := \{m \in M \mid \ \forall g \in A \ g \, \mathrm{I} \, m\}$
- $M \supseteq B \mapsto B' := \{g \in G \mid \ \forall m \in B \ g \, \mathrm{I} \, m\}$
- **formal concept** := a pair (A, B) with $A' = B$ and $B' = A$.
- A :=**extent** and B := **intent** of the concept (A, B).
- $\mathfrak{B}(\mathbb{K})$:= set of all formal concepts of $\mathbb{K}$.
- $G \supseteq A \mapsto A''$ and $M \supseteq B \mapsto B''$ are closure operators.
- $\mathrm{Ext}(\mathbb{K})$:= set of all extents of $\mathbb{K}$ and $\mathrm{Int}(\mathbb{K})$:= set all intents.

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$(A, B) \leq (C, D) :\iff A \subseteq C \qquad (\iff D \subseteq B).$

$\underline{\mathfrak{B}}(G, M, I) := (\mathfrak{B}(G, M, I), \leq)$ is a complete lattice in which infimum and supremum are given by:

$$\bigwedge_{t\in T} (A_t, B_t) = \left(\bigcap_{t\in T} A_t, \left(\bigcup_{t\in T} B_t \right)'' \right)$$

$$\bigvee_{t\in T} (A_t, B_t) = \left(\left(\bigcup_{t\in T} A_t \right)'', \bigcap_{t\in T} B_t \right).$$

$\underline{\mathfrak{B}}(G, M, I)$ is called the concept lattice of the context (G, M, I).

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A complete lattice L is isomorphic to a concept lattice $\underline{\mathfrak{B}}(G, M, I)$ iff there are mappings $\tilde{\gamma} : G \to L$ and $\tilde{\mu} : M \to L$ such that $\tilde{\gamma}(G)$ is supremum-dense in L, $\tilde{\mu}(M)$ is infimum-dense in L and for all $g \in G$ and $m \in M$

$$gIm \iff \tilde{\gamma}(g) \leq \tilde{\mu}(m).$$

In particular $L \cong \underline{\mathfrak{B}}(L, L, \leq)$.

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- A special context: $\mathbb{K} := (X, X, \neq)$.
- concepts of $(X, X, \neq)$ are pairs $(A, X \setminus A)$ with any $A \subset X$.
- The negation of (A, B) is then (B, A).
- $\underline{\mathfrak{B}}(X, X, \neq)$ is a Boolean algebra isomorphic to $\mathcal{P}(X)$.

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Two options for negation:

- Relax the definition of concept to semi- pre- and protoconcepts $\rightsquigarrow$ double Boolean algebras.
- Replace the complements with their closures $\rightsquigarrow$ weakly dicomplemented lattices.

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Semi-, pre- and protoconcepts

Let $\mathbb{K} := (G, M, I)$ be a formal context. $A \subseteq G$ and $B \subseteq M$.

- A **semiconcept** is a pair (A, B) with $A = B'$ or $B = A'$.
- A **preconcept** is a pair (A, B) with $A \subseteq B'$ (iff $B \subseteq A'$)
- A **protoconcept** is a pair (A, B) with $A'' = B'$ (iff $B' = A''$)
- $\mathfrak{P}(\mathbb{K}) :=$ set of all protoconcepts of the context $\mathbb{K}$
- Logical operations:

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- $\mathfrak{P}(\mathbb{K})_{\sqcap} := \{(A, A') \mid A \subseteq G\}$ $\sqcap$-semiconcepts.
- $\mathfrak{P}(\mathbb{K})_{\sqcup} := \{(B', B) \mid B \subseteq M\}$ $\sqcup$-semiconcepts.
- $\mathfrak{H}(\mathbb{K}) := \mathfrak{P}(\mathbb{K})_{\sqcap} \cup \mathfrak{P}(\mathbb{K})_{\sqcup}$ semiconcepts
- $\mathfrak{B}(\mathbb{K}) = \mathfrak{P}(\mathbb{K})_{\sqcap} \cap \mathfrak{P}(\mathbb{K})_{\sqcup}$ concepts

- Further operations:

  $x \oplus y := (x^{\lhd} \sqcap y^{\lhd})^{\lhd}$, and $x \odot y := (x^{\rhd} \sqcup y^{\rhd})^{\rhd}$,

  $\mathtt{1} := \bot^{\lhd}$ and $\mathtt{0} := \top^{\rhd}$.

$\underline{\mathfrak{P}}(\mathbb{K})_{\sqcap} := (\mathfrak{P}(\mathbb{K})_{\sqcap}, \sqcap, \oplus, ^{\lhd}, \bot, \mathtt{1})$ is a Boolean algebra isomorphic to $\underline{\mathcal{P}}(G)$ and $\underline{\mathfrak{P}}(\mathbb{K})_{\sqcup} := (\mathfrak{P}(\mathbb{K})_{\sqcup}, \odot, \sqcup, ^{\rhd}, \mathtt{0}, \top)$ a Boolean algebra dual isomorphic to $\mathcal{P}(M)$.

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- Further operations:

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  $\mathtt{1} := \bot^{\lhd}$ and $\mathtt{0} := \top^{\rhd}$.

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$$
\begin{array}{llll}
(1) & x \sqcap y = y \sqcap x, & (1') & x \sqcup y = y \sqcup x, \\
(2) & x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z, & (2') & x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z, \\
(3) & x \sqcap (x \sqcup y) = x \sqcap x, & (3') & x \sqcup (x \sqcap y) = x \sqcup x, \\
(4) & x \sqcap (x \oplus y) = x \sqcap x, & (4') & x \sqcup (x \odot y) = x \sqcup x, \\
(5) & (x \sqcap x) \sqcap y = x \sqcap y, & (5') & (x \sqcup x) \sqcup y = x \sqcup y, \\
(6) & x \sqcap (y \oplus z) = & (6') & x \sqcup (y \odot z) = \\
 & (x \sqcap y) \oplus (x \sqcap z), & & (x \sqcup y) \odot (x \sqcup z), \\
(7) & (x \sqcap y)^{\lhd\lhd} = x \sqcap y, & (7') & (x \sqcup y)^{\rhd\rhd} = x \sqcup y, \\
(8) & (x \sqcap x)^{\lhd} = x^{\lhd}, & (8') & (x \sqcup x)^{\rhd} = x^{\rhd}, \\
(9) & x \sqcap x^{\lhd} = \bot, & (9') & x \sqcup x^{\rhd} = \top, \\
(10) & \bot^{\lhd} = \top \sqcap \top, & (10') & \top^{\rhd} = \bot \sqcup \bot, \\
(11) & \top^{\lhd} = \bot, & (11') & \bot^{\rhd} = \top, \\
(13) & x \sqcap x \leq (x \sqcap y) \sqcup (x \sqcap y^{\lhd}) & (13') & x \sqcup x \geq (x \sqcup y) \sqcap (x \sqcup y^{\rhd})
\end{array}
$$

$$
(12) \quad (x \sqcap x) \sqcup (x \sqcap x) = (x \sqcup x) \sqcap (x \sqcup x).
$$

Double Boolean algebras

An algebra $(D, \sqcap, \sqcup, ^{\lhd}, ^{\rhd}, \bot, \top)$ of type $(2, 2, 1, 1, 0, 0)$ that satisfies (1) to (13) and (1') to (13') a **double Boolean algebra**
A double Boolean algebra is called **pure** if it satisfies

(14) $x \sqcap x = x$ or $x \sqcup x = x$

Notation:

- $x_{\sqcap} := x \sqcap x, \quad D_{\sqcap} := \{x_{\sqcap} \mid x \in D\}$
- $x_{\sqcup} := x \sqcup x, \quad D_{\sqcup} := \{x_{\sqcup} \mid x \in D\}$

$\underline{D}_{\sqcap} := (D_{\sqcap}, \sqcap, \oplus, ^{\lhd}, \bot, \imath)$ and $\underline{D}_{\sqcup} := (D_{\sqcup}, \odot, \sqcup, ^{\rhd}, o, \top)$ are Boolean algebras, where $x \oplus y := (x^{\lhd} \sqcap y^{\lhd})^{\lhd}$, $x \odot y := (x^{\rhd} \sqcup y^{\rhd})^{\rhd}$, $\imath := \bot^{\lhd}$ and $o := \top^{\rhd}$

Let D be a double Boolean algebra D and $x, y, a \in D$

- $x \sqsubseteq y :\iff x \sqcap y = x \sqcap x$ and $x \sqcup y = y \sqcup y$
- $x \sqsubseteq y \iff x_\sqcap \sqcap y_\sqcap = x_\sqcap$ and $x_\sqcup \sqcup y_\sqcup = y_\sqcup$.
- $x \sqsubseteq y \iff x_\sqcap \leq y_\sqcap$ and $x_\sqcup \leq y_\sqcup$
- $x \sqcap y \sqsubseteq x, y \sqsubseteq x \sqcup y$
- $x \sqsubseteq y$ implies $x \sqcap a \sqsubseteq y \sqcap a$ and $x \sqcup a \sqsubseteq y \sqcup a$

- A Filter is nonempty subset F of D such that

$$x, y \in F \implies x \sqcap y \in F, \text{ and } x \in F,\ y \in D,\ x \sqsubseteq y \implies y \in F.$$

- An Ideal is a nonempty subset I of D satisfying

$$x, y \in I \implies x \sqcup y \in I, \text{ and } x \in I,\ y \in D,\ x \sqsupseteq y \implies y \in I.$$

Prime Ideal Theorem

Let D be a double Boolean algebra. A filter F is called proper if $F \neq D$, and **primary** if it is proper and satisfies $x \in F$ or $x^{\triangleleft} \in F$ for all $x \in D$. Dually are defined **primary ideals**. $\mathcal{F}_{pr}(D)$ denotes the set of primary filters and $\mathcal{I}_{pr}(D)$ the set of primary ideals of D.

Theorem (Prime ideal theorem)

Let D be a double Boolean algebra, F a filter and I an ideal such that $F \cap I = \emptyset$.

1. *There exists a primary filter G and a primary ideal J with $F \subseteq G$, $I \subseteq J$ and $G \cap J = \emptyset$.*
2. $x \sqcap x \not\sqsubseteq y \sqcup y \implies \exists$ *a primary filter G with $x \in G$ and $y \notin G$.*
3. *Nontrivial double Boolean algebras have primary filters / ideals.*

Weak Negation and weak opposition

To formalize a negation two operations are introduced:

Definition

Let $\mathbb{K}$ be a context and (A, B) a formal concept of $\mathbb{K}$. We define

its **weak negation** by $\quad (A, B)^{\triangle} := \left((G \setminus A)'', (G \setminus A)'\right)$

and its **weak opposition** by $\quad (A, B)^{\nabla} := \left((M \setminus B)', (M \setminus B)''\right).$

$\mathfrak{A}(\mathbb{K}) := \left(\mathfrak{B}(\mathbb{K}); \wedge, \vee, ^{\triangle}, ^{\nabla}, 0, 1\right)$ is called the **concept algebra** of the formal context $\mathbb{K}$, where $\wedge$ and $\vee$ denote the meet and the join operations of the concept lattice.

Weak dicomplementation

A **weakly dicomplemented lattice** is a bounded lattice L equipped with two unary operations $^\triangle$ (**weak complementation**) and $^\nabla$ (**dual weak complementation**), such that for all $x, y \in L$:

(1) $x^{\triangle\triangle} \leq x$,

(2) $x \leq y \implies x^\triangle \geq y^\triangle$,

(3) $(x \wedge y) \vee (x \wedge y^\triangle) = x$,

(1') $x^{\nabla\nabla} \geq x$,

(2') $x \leq y \implies x^\nabla \geq y^\nabla$,

(3') $(x \vee y) \wedge (x \vee y^\nabla) = x$.

1. $x^\triangle :\equiv$ **weak complement** of x
2. $x^\nabla :\equiv$ **dual weak complement** of x
3. $(x^\triangle, x^\nabla) :\equiv$ **weak dicomplement** of x
4. $(^\triangle, ^\nabla) :\equiv$ **weak dicomplementation** on L
5. $(L, \wedge, \vee, ^\triangle, 0, 1) :\equiv$ **weakly complemented lattice**
6. $(L, \wedge, \vee, ^\nabla, 0, 1) :\equiv$ **dual weakly complemented lattice**

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2. x^{∇} :≡ dual weak complement of x
3. $(x^{\triangle}, x^{\nabla})$:≡ weak dicomplement of x
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Weak dicomplementation: some properties

1. $x \vee x^{\triangle} = 1$
2. $x \wedge x^{\nabla} = 0$
3. $0^{\triangle} = 1 = 0^{\nabla}$
4. $1^{\triangle} = 0 = 1^{\nabla}$
5. $x^{\nabla} \leq x^{\triangle}$
6. $(x \wedge y)^{\triangle} = x^{\triangle} \vee y^{\triangle}$
7. $(x \vee y)^{\nabla} = x^{\nabla} \wedge y^{\nabla}$
8. $x^{\triangle\triangle\triangle} = x^{\triangle}$
9. $x^{\nabla\nabla\nabla} = x^{\nabla}$
10. $x^{\triangle\nabla} \leq x^{\triangle\triangle} \leq x \leq x^{\nabla\nabla} \leq x^{\nabla\triangle}$
11. $x \mapsto x^{\triangle\triangle}$ is a kernel operator on L
12. $x \mapsto x^{\nabla\nabla}$ is a closure operator on L

Examples of weak dicomplementations

- Boolean algebras: $(B, \wedge, \vee, \bar{\ }, 0, 1) \rightsquigarrow (B, \wedge, \vee, \bar{\ }, \bar{\ }, 0, 1)$.
- Trivial weak dicomplementation on bounded lattices:

  Define $(1,1)$, $(0,0)$ and $(1,0)$ as the weak dicomplement of 0, 1 and of each $x \notin \{0,1\}$, respectively.
- Distributive double p-algebras $(L, \wedge, \vee, ^+, ^*, 0, 1)$:
- On finite lattice L:

  $J(L) :=$ join irreducible elts and $M(L) :=$ meet irreducible elts
- $x^{\triangle} := \bigvee\{a \in J(L) \mid a \not\leq x\}$ and $x^{\triangledown} := \bigwedge\{m \in M(L) \mid m \not\geq x\}$.
- For $G \supseteq J(L)$ and $H \supseteq M(L)$, define ${}^{\triangle_G}$ and ${}^{\triangledown_H}$ by

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- Boolean algebras: $(B, \wedge, \vee, \bar{}, 0, 1) \rightsquigarrow (B, \wedge, \vee, \bar{}, \bar{}, 0, 1)$.
- Trivial weak dicomplementation on bounded lattices:

  Define $(1, 1)$, $(0, 0)$ and $(1, 0)$ as the weak dicomplement of 0, 1 and of each $x \notin \{0, 1\}$, respectively.
- Distributive double p-algebras $(L, \wedge, \vee, ^+, ^*, 0, 1)$:
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Let h be a closure operator on X and k a kernel operator on Y.
$A^{\triangle_h} := h(X \setminus A)$, $B^{\nabla_k} := k(Y \setminus B)$ for $A \subseteq X$ and $B \subseteq Y$.

(i) $(h\mathcal{P}(X), \cap, \vee^h, {}^{\triangle_h}, h\emptyset, X)$, with $A_1 \vee^h A_2 := h(A_1 \cup A_2)$, is a weakly complemented lattice.

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Representation problem

equational axiomatization: Find equations that generate the equational theory of concept algebras.

strong representation: Describe weakly dicomplemented lattices that are isomorphic to concept algebras.

isomorphism problem: Given a complete weakly dicomplemented lattice L, is there a context $\mathbb{K}(L)$ such that $\mathfrak{A}(\mathbb{K}(L)) \cong L$?

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Theorem

The mapping

$$\begin{array}{rccc} \varphi: & L & \to & \underline{\mathfrak{B}}(\mathbb{K}(L)) \\ & x & \mapsto & (\mathfrak{F}_x, \mathfrak{I}_x) \end{array}$$

is a lattice embedding.

Problem: is φ a weakly dicomplemented lattice embedding?

Dreamlike embedding

What about the weak operations?

- $\mathfrak{I}_{x^\triangle} \subseteq (\mathfrak{F}_{pr}(L) \setminus \mathfrak{F}_x)'$
- $\mathfrak{F}_{x^\nabla} \subseteq (\mathfrak{I}_{pr}(L) \setminus \mathfrak{I}_x)'$

Thus $\varphi(x^\nabla) \leq \varphi(x)^\nabla \leq \varphi(x)^\triangle \leq \varphi(x^\triangle)$.

Where is the problem?

Let I be a primary ideal such that $I \not\ni x^\triangle$. If $x \notin I$ but $x^\triangle \in Ideal(I \cup \{x \wedge x^\triangle\})$, **is there a primary filter** F **such that** $x \notin F$ **and** $F \cap I = \emptyset$?

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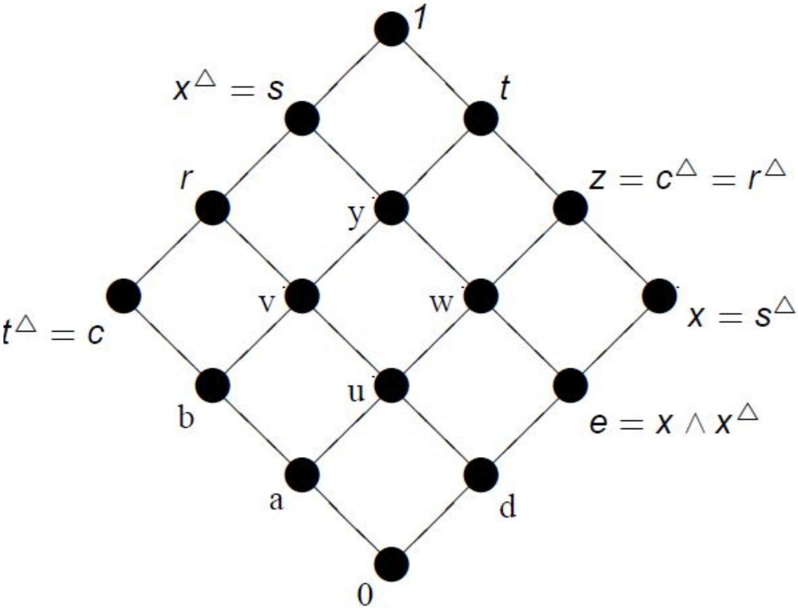

1
$x^{\triangle} = s$
t
r
$z = c^{\triangle} = r^{\triangle}$
y
$t^{\triangle} = c$
v
w
$x = s^{\triangle}$
b
u
$e = x \wedge x^{\triangle}$
a
d
0

Special cases:

- If L is a Boolean algebra, then $\mathfrak{F}_{\text{pr}}(L)$ is the set of ultrafilters of L and is one-to-one to $\mathfrak{I}_{\text{pr}}(L)$. The context $(\mathfrak{F}_{\text{pr}}(L), \mathfrak{I}_{\text{pr}}(L), \square)$ and $(\mathfrak{F}_{\text{pr}}(L), \mathfrak{F}_{\text{pr}}(L), \neq)$ are isomorphic and their concept lattice is a complete and atomic Boolean algebra isomorphic to $\mathcal{P}(\mathfrak{F}_{\text{pr}})$.

  L embeds into the concept algebra of $\mathbb{K}^{\triangle}_{\triangledown}(L)$.

Corollary (Stone '36)

Each Boolean algebra embeds into a field of sets.

- Finite and distributive lattices:

  The concrete representation is ok. But the proof uses combinatorial arguments and is based on a different approach.

LK (2009)

An algebra $(L, \wedge, \vee, ^{\triangle})$ of type $(2, 2, 1)$ is a Boolean algebra iff

$$(L, \wedge, \vee) \text{ is a non empty lattice and} \tag{1}$$
$$(x \wedge y) \vee (x \wedge y^{\triangle}) = (x \vee y) \wedge (x \vee y^{\triangle}) \text{ for all } x, y \in L. \tag{2}$$

$$x \geq (x \wedge y) \vee (x \wedge y^{\triangle}) = (x \vee y) \wedge (x \vee y^{\triangle}) \geq x \tag{3}$$
$$\implies x = (x \wedge y) \vee (x \wedge y^{\triangle}) \tag{4}$$
$$\implies x = (x \vee y) \wedge (x \vee y^{\triangle}) \tag{5}$$

Each concept x decomposes into a negative and a positive part with respect to any other concept y, as well for the disjunction as for the conjunction.

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Proof: $(x \wedge y) \vee (x \wedge y^{\triangle}) = (x \vee y) \wedge (x \vee y^{\triangle})$

1. Each nonempty lattice satisfying (1)-(3) is bounded.
   Take $x \in L$ and set $1 := x \vee x^{\triangle}$ and $0 := 1^{\triangle}$.

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- For $c \in L \setminus \{0, 1\}$, we have $[c, 1] \cong [0, c^{\triangle}]$, since

$$\begin{array}{rcl} u_{c^{\triangle}} : [c, 1] & \to & [0, c^{\triangle}] \\ x & \mapsto & x \wedge c^{\triangle} \end{array} \quad \text{and} \quad \begin{array}{rcl} v_c : [0, c^{\triangle}] & \to & [c, 1] \\ x & \mapsto & x \vee c \end{array}$$

  are order preserving and inverse of each other.

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   - The maps

$$\begin{array}{rcccl} f_1 : L & \to & [c^{\triangle}, 1] & \to & [0, c^{\triangle\triangle}] = [0, c] \\ x & \mapsto & x \vee c^{\triangle} & \mapsto & (x \vee c^{\triangle}) \wedge c = x \wedge c \end{array}$$

     and

$$\begin{array}{rcccl} f_2 : L & \to & [c, 1] & \to & [0, c^{\triangle}] \\ x & \mapsto & x \vee c & \mapsto & (x \vee c) \wedge c^{\triangle} = x \wedge c^{\triangle} \end{array}$$

     are lattice homomorphisms.

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- Set $\theta_1 := \ker f_1$ and $\theta_2 := \ker f_2$. Then $\theta_1 \cap \theta_2 = \Delta$.

$$
\begin{aligned}
(x, y) \in \theta_1 \cap \theta_2 &\implies x \wedge c = y \wedge c \text{ and } x \wedge c^{\triangle} = y \wedge c^{\triangle} \\
&\implies \underbrace{(x \wedge c) \vee (x \wedge c^{\triangle})}_{x} = \underbrace{(y \wedge c) \vee (y \wedge c^{\triangle})}_{y} \\
&\implies (x, y) \in \Delta.
\end{aligned}
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3. $(L, \wedge, \vee, ^{\triangle}, 0, 1)$ is a Boolean algebra iff $(L, \wedge, \vee)$ is a non empty lattice in which i) $x^{\triangle\triangle} = x$, (ii) $x \leq y \implies x^{\triangle} \geq y^{\triangle}$ and (iii) $(x \wedge y) \vee (x \wedge y^{\triangle}) = x = (x \vee y) \wedge (x \vee y^{\triangle})$.

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(iii) $x \geq (x \wedge y) \vee (x \wedge y^{\triangle}) = (x \vee y) \wedge (x \vee y^{\triangle}) \geq x$ implies $(x \wedge y) \vee (x \wedge y^{\triangle}) = x = (x \vee y) \wedge (x \vee y^{\triangle})$.

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   (i) $x = (x \vee y) \wedge (x \vee y^{\triangle}); \implies y \wedge y^{\triangle} = 0;$
   $x = (x \wedge x^{\triangle}) \vee (x \wedge x^{\triangle\triangle}) = 0 \vee (x \wedge x^{\triangle\triangle}) = x \wedge x^{\triangle\triangle}$
   Hence $x \leq x^{\triangle\triangle}$.
   $x = (x \wedge y) \vee (x \wedge y^{\triangle}); \implies y \vee y^{\triangle} = 1;$
   $x = (x \vee x^{\triangle}) \wedge (x \vee x^{\triangle\triangle}) = 1 \wedge (x \vee x^{\triangle\triangle}) = x \vee x^{\triangle\triangle}$
   Hence $x \geq x^{\triangle\triangle}$. Therefore $x = x^{\triangle\triangle}$.

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(ii) Let $x \leq y$.
$x \vee x^{\triangle} = 1 \implies y \vee x^{\triangle} = 1$. Thus
$x^{\triangle} = (x^{\triangle} \vee y^{\triangle}) \wedge (x^{\triangle} \vee y^{\triangle\triangle}) = (x^{\triangle} \vee y^{\triangle}) \wedge (x^{\triangle} \vee y) = x^{\triangle} \vee y^{\triangle}$
and $x^{\triangle} \geq y^{\triangle}$.

Theorem (A new axiom for Boolean algebras)

Boolean algebras are exactly nonempty lattices $(L, \wedge, \vee)$ *with a unary operation* $^{\triangle}$ *satisfying:*

$$(x \wedge y) \vee (x \wedge y^{\triangle}) = (x \vee y) \wedge (x \vee y^{\triangle})$$

for all $x, y \in L$.