A meeting between two theories : the development of Boolean Algebra and Lattice Theory in the 1930s

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GBMS Theme 2 : From Boole's Algebra of Logic to Boolean Algebra, and Beyond

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Boolean Algebra and Lattice Theory

Two new theories : Lattice Theory

"It was Garrett Birkhoff's work in the mid-thirties that started the general development of lattice theory." (G. Grätzer, 1971)

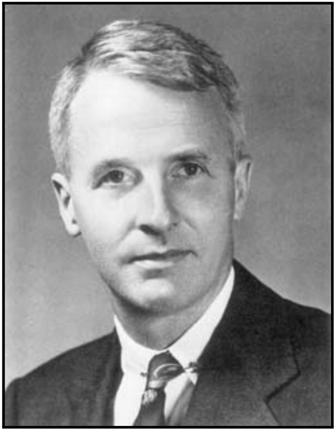
"Birkhoff and Ore were the leading figures behind the reelaboration of lattice theory in the 1930s." (L. Corry, 1996)

Two new theories : Boolean algebra

"There are two aspects in the theory of Boolean algebras (...). Fundamental theorems in both directions are due to M. H. Stone whose papers have opened a new period in the development of the theory." (R. Sikorski, 1960)

"Only with the work of Marshall Stone and Alfred Tarski in the 1930s (...) did Boolean algebra free itself completely from the bonds of logic and become a modern mathematical discipline." (S. Givant, 2009)

Two leading figures



Garrett Birkhoff

Marshall H. Stone

- A case study on the establishment of a mathematical theory
- Between the first paper of Birkhoff on lattice (1933) and the publication of *Lattice Theory* (1940)
- In the context of the *American Mathematical Society*

- What can we say about Boolean Algebra and Lattice as mathematical theories ?

1. *Comparison* : What do the theories have in common ? What are their differencies ?

2. *Crossing* : How do the theories interact ? What can we say from one focusing on the other ?

Comparing : two theories ?

Corpus : from abstract to publication

tico-Physical Society, vol. 9, pp. 155–159). The general theory is applied for the derivation of the distribution function of means, standard deviations, simultaneous distribution of means and standard deviations, correlations, Student's ratio, χ^2, etc. for samples of $n(n \geqq 1)$ drawn from various parent distributions. (Received January 10, 1933.)

86. Professor M. H. Stone: *On the structure of Boolean algebras.*

This paper presents a theory of ideals and homeomorphisms for Boolean algebras and related systems which may be described as unitless Boolean algebras. The analogy with ideal theory in abstract rings (see van der Waerden's *Moderne Algebra*) is brought out by means of new sets of postulates for these algebras. Under the assumption that such an algebra A can be well-ordered, the existence and properties of prime ideals in A are established. Every ideal is shown to be the product of all its prime ideal divisors (it may be the product of fewer). The algebra A is shown to be isomorphic with an algebra A^* whose elements are subsets of the set E of all prime ideals in A. Each set in A^* can be regarded as a neighborhood (in the sense of Hausdorff) of every prime ideal belonging to it. When E is thus topologized, algebraic properties of A are interpretable as topological properties of E and conversely. In particular, the algebras homeomorphic with A are determined (up to isomorphisms) by means of the topological space E. It is of interest to note that all analogues of the maximal and minimal conditions so frequently postulated by algebraists lead in the present instance to finite algebras. (Received January 24, 1933.)

87. Dr. E. F. Beckenbach (National Research Fellow): *Bloch's theorem for minimal surfaces.*

There exists a positive absolute constant B with the following property. Let the unit circle be mapped conformally on a minimal surface with unit area deformation ratio at the origin; then on the minimal surface there is an open geodesic circle of radius at least B, containing no singular points, which is the one-to-one map of a portion of the unit circle. (Received January 9, 1933.)

88. Professor L. E. Ward: *A third-order irregular boundary value problem and the associated series.*

This paper is concerned with the characteristic functions defined by the differential system $u''' + [\rho^3 + r(x)]u = 0$, $\alpha_{12}u''(0) + \alpha_{11}u'(0) + \alpha_{10}u(0) = 0$, $\alpha_{22}u''(0) + \alpha_{21}u'(0) + \alpha_{20}u(0) + \beta_{22}u''(\pi) + \beta_{21}u'(\pi) + \beta_{20}u(\pi) = 0$, $\alpha_{31}u'(0) + \alpha_{30}u(0) = 0$, where the α's and β's are real costants, the determinant

Abstract submited to the *AMS*

Corpus : from abstract to publication

18. *On Young's definition of an algebra*, by Dr. L. E. Bush. (Abstract No. 39–3–82–*t*.)
19. *On a generalization of certain theorems on algebras with a modulus*, by Dr. L. E. Bush. (Abstract No. 39–3–83–*t*.)
20. *The representation of integers as sums of pyramidal numbers*, by Dr. R. D. James (National Research Fellow). (Abstract No. 39–3–84–*t*.)
21. *An application of characteristic functions to statistics* (preliminary report), by Mr. Solomon Kullback. (Abstract No. 39–3–85–*t*.)
22. *On the structure of Boolean algebras*, by Professor M. H. Stone. (Abstract No. 39–3–86–*t*.)
23. *Bloch's theorem for minimal surfaces*, by Dr. E. F. Beckenbach (National Research Fellow). (Abstract No. 39–3–87–*t*.)
24. *A third-order irregular boundary value problem and the associated series*, by Professor L. E. Ward. (Abstract No. 39–3–88–*t*.)
25. *Notes on the theory and application of Fourier transforms*, I–II, by Dr. R. E. A. C. Paley and Professor Norbert Wiener. (Abstract No. 39–3–89–*t*.)
26. *On the decomposition of prime ideals in relative icosahedron fields*, by Dr. R. M. Gut (Rockefeller Foundation Fellow). (Abstract No. 39–3–90.)
27. *A Jordan space-curve which bounds no finite simply-connected area*, by Professor Jesse Douglas. (Abstract No. 39–3–91.)
28. *On the division transformation for matric polynomials*, by Mr. M. M. Flood. (Abstract No. 39–3–92.)
29. *The uniformly loaded thick rectangular plate with at least two opposite edges supported*, by Professor C. A. Garabedian. (Abstract No. 39–3–93.)
30. *A study of certain dynamical systems with applications to the generalized double pendulum*, by Dr. G. B. Price. (Abstract No. 39–3–94.)

22. *On the structure of Boolean algebras*, by Professor M. H. Stone. (Abstract No. 39–3–86–*t*.)

Presentation following the abstract

Corpus : from abstract to publication

Sokolnikoff, E. S., and Sokolnikoff, I. S. *Note on a resolution of linear differential systems*. Read April 14, 1933. Proceedings of the Edinburgh Mathematical Society, (2), vol. 4, No. 1, pp. 36–40; March, 1934.
Sokolnikoff, I. S. See Sokolnikoff, E. S.
Sperner, E. *Über die fixpunktfreie Abbildungen der Ebene*. Read Sept. 2, 1932. Hamburger Mathematische Einzelschriften, No. 14, 1933. 47 pp.
Steenrod, N. E. *Characterizations of certain finite curve-sums*. Read Dec. 27, 1933. American Journal of Mathematics, vol. 56, No. 4, pp. 558–568; Oct., 1934.
—— *Finite arc-sums*. Read April 14, 1933. Fundamenta Mathematicae, vol. 23, pp. 38–53; 1934.
Stelford, N. See Simmons, H. A.
Stephens, R. *Note on a problem of Fréchet*. Read Oct. 28, 1933. This Bulletin, vol. 40, No. 2, pp. 65–68; Feb., 1934.
—— *Continuous transformations of finite spaces*. Read Sept. 2, 1932. Tôhoku Mathematical Journal, vol. 39, No. 1, pp. 98–106; April, 1934.
Stone, M. H. *Boolean algebras and their applications to topology*. Read Feb. 25, 1933. Proceedings of the National Academy of Sciences, vol. 20, No. 3, pp. 197–202; March, 1934.
Synge, J. L. *On the deviation of geodesics and null-geodesics, particularly in relation to the properties of spaces of constant curvature and indefinite line-element*. Read Dec. 27, 1933. Annals of Mathematics, (2), vol. 35, No. 4, pp. 705–713; Oct., 1934.
Tamarkin, J. D. See Hille, E.
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Thielman, H. P. *On the invariance of a generalized Gramian in a Riemannian function space*. Read April 13, 1933. American Journal of Mathematics, vol. 56, No. 3, pp. 438–444; July, 1934.
—— *Note on the use of fractional integration of Bessel functions*. Read April 7, 1934. This Bulletin, vol. 40, No. 10, pp. 695–698; Oct., 1934.
Thomas, J. M. *A lower limit for the species of a Pfaffian system*. Read Oct. 28, 1933. Proceedings of the National Academy of Sciences, vol. 19, No. 10, pp. 913–914; Oct., 1933.
—— *Riquier's existence theorems*. Read Nov. 28, 1931. Annals of Mathematics, (2), vol. 35, No. 2, pp. 306–311; April, 1934.
—— *An existence theorem for generalized Pfaffian systems*. Read Dec. 1, 1933. This Bulletin, vol. 40, No. 4, pp. 309–315; April, 1934.
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Corpus : from abstract to publication

BOOLEAN ALGEBRAS AND THEIR APPLICATION TO TOPOLOGY[1]

By M. H. Stone

Division of Mathematics, Harvard University

Read before the Academy, Wednesday, November 22, 1933

The purpose of this note is to present the essential features of two papers now in preparation for submission to the Transactions of the American Mathematical Society.

A Boolean algebra may be defined by a set of postulates in terms of (logical) addition and (logical) multiplication as undefined operations.[2] It is convenient, in algebraic discussions at least, to admit the customarily excluded case in which the algebra contains just one element. By analogy with the theory of abstract rings,[3] we are led to consider a theory of ideals and homomorphisms for Boolean algebras. An ideal $\mathfrak{a}$ in a Boolean algebra A is a class of elements belonging to A with the properties:

(1) if a and b are in $\mathfrak{a}$, their sum $a + b$ is in $\mathfrak{a}$;
(2) if a is in $\mathfrak{a}$ and b is in A, their product ab is in $\mathfrak{a}$.

Two elements b, c in A are said to be congruent (mod $\mathfrak{a}$) if their symmetric difference[4] is an element in the ideal $\mathfrak{a}$. An ideal $\mathfrak{a}$ determines a partition of the elements belonging to A into disjoint subclasses $\mathfrak{k}$ of elements mutually congruent (mod $\mathfrak{a}$) and does so in such a manner that the subclasses $\mathfrak{k}$ constitute a Boolean algebra A^* when the sum (product) of $\mathfrak{k}_1$ and $\mathfrak{k}_2$ is defined as the class $\mathfrak{k}_3$ which contains the sum (product) of some element in $\mathfrak{k}_1$ and some element in $\mathfrak{k}_2$; this algebra A^* is called the quotient of A by $\mathfrak{a}$ and is denoted by $A/\mathfrak{a}$. The fundamental theorem concerning homomorphisms may now be stated as follows:

Publication

Corpus : from abstract to publication

- A corpus built on keywords (“Boolean”, “lattice” and “structure”)
- Two subcorpuses : *BA* and *Lattice*

Appropriate subjects for publication

Number of ...	*BA*	*Lattice*	Intersection	Total *AMS*
Abstracts :	53	83	17	8085
Publications :	33 (62%)	50 (60%)	12	2493 (31%)

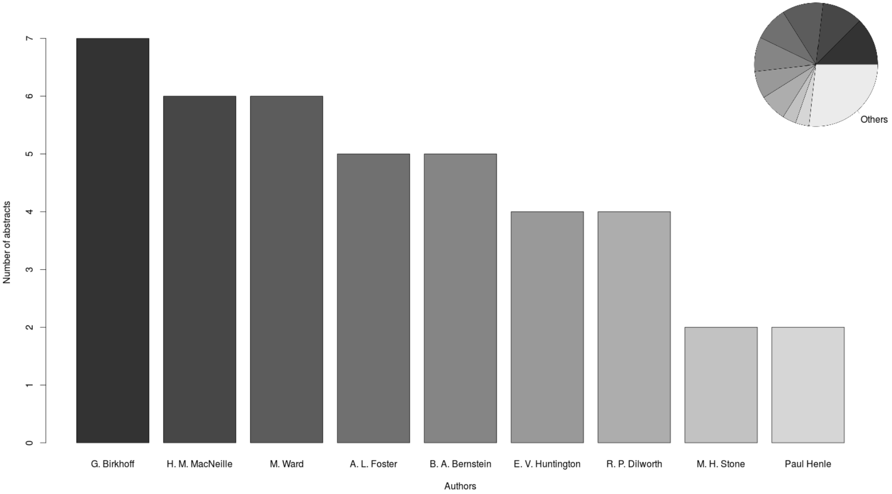
- More abstracts on lattices but the same proportion of resulting publications
- A proportion of publications greater than the average
- A few texts belonging to both corpuses

Comparing the authors

Authors that published the most abstracts in the two corpuses

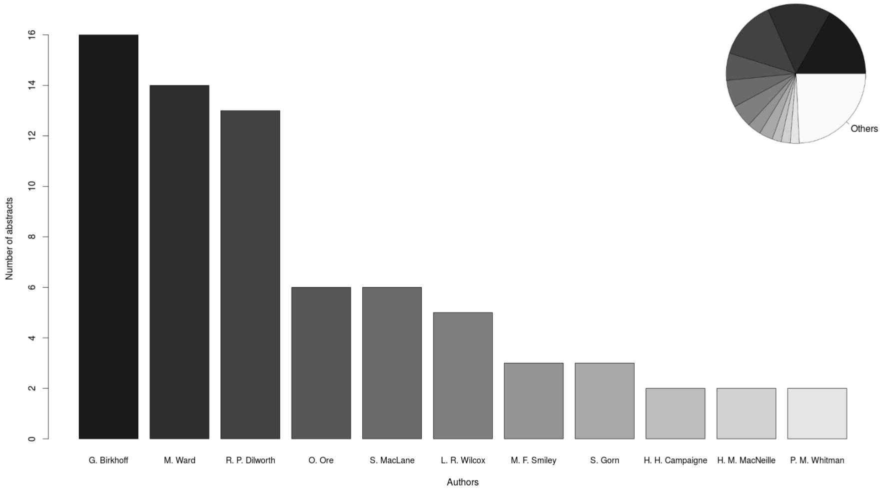
- Who are the main authors in both corpus ?
- Is there two different communities ?

Authors : the *BA* corpus



(Authors who published more that 1 abstract)

Authors : the *L* corpus



(Authors who published more that 1 abstract)

Authors

- A small core of authors submitting a lot of abstracts in each corpus
- Three shared authors : G. Birkhoff, M. Ward, R. P. Dilworth
- Stone barely appears in the corpus on Boolean Algebras

Références from the *BA* corpus

What do we learn from the material cited by the *BA* corpus ?

- The most cited article is Birkhoff's 1933 paper on lattice theory
- Then : 2 papers by Huntington
- 7 papers by Stone are cited 17 times by 11 publications

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  - $\rightarrow$ Meeting of the two corpuses
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- 7 papers by Stone are cited 17 times by 11 publications
  - → Importance of Stone's work in a particular context

Stone’s algebraic theory of Boolean algebras

“In an earlier paper we have developed an abstract theory of Boolean algebras and their representations by algebras of classes.” (Stone, 1937)

“For algebraic developments of Boolean rings, particularly in the light of their relation with the rings and ideals of classical abstract algebra, see M. H. Stone, The Theory of Representations for Boolean Algebras” (Koopman, 1940)

Stone’s algebraic theory of Boolean algebras

“Stone is able to employ modern algebraic methods in his study of Boolean algebra. To my mind his thoroughgoing papers constitute the most considerable advance in our understanding of Boolean algebras since Boole’s own work.” (G. D. Birkhoff, 1938)

On comparison

What did we learn from comparison ?

- Similar status : a limited number of texts and a small core of authors
- Common authors and common interests
- A particular trend : algebraic theory of Boolean algebras

On comparison

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  - → Specific work and not very general subjects
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  - → Specific work and not very general subjects
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  - → Two different theories ?
- A particular trend : algebraic theory of Boolean algebras

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  - → Specific work and not very general subjects
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  - → Two different theories ?
- A particular trend : algebraic theory of Boolean algebras
  - → A meeting point

Crossing : Birkhoff and Stone on representation

- Focus on the particular algebraic work highlited by the comparison
- A theorem on both Boolean algebra and Lattices

Stone's representation theorem

"By analogy with the theory of abstract rings, we are led to consider a theory of ideals and homomorphisms for Boolean algebras"

- ideal, prime ideal, quotient algebra
- homomorphism theorem
- three theorems linking together Boolean algebras and specifics spaces
  - $\rightarrow$ The purpose is to study abstract topology

Stone, *Boolean Algebras and their application to topology*, 1934

Stone's representation theorem

Theorem

Let A be an arbitrary Boolean algebra, $\mathfrak{E}$ the set of all prime ideals in A, $\mathfrak{E}(a)$ the set of all prime ideals in A which do not contain the element a, and $B(A)$ the algebra of the sets $\mathfrak{E}(a)$ with sum and product defined as union and intersection, respectively. Then the correspondence $a \leftrightarrow \mathfrak{E}(a)$ defines an isomorphism of A and $B(A)$.

"A more general theorem was (...) obtained and published by Garrett Birkhoff"

Stone, *Boolean Algebras and their application to topology*, 1934

Birkhoff's representation theorem

Theorem

The algebras satisfying Axioms I–IV and VI are collectively identical with the algebras of classes of sets of points with respect to finite joins and meets.

Birkhoff, *On the combination of subalgebras*, 1933

Boolean rings

"In a previous communication, we have sketched an analogy between Boolean algebras and abstract rings (...). In the present note we shall show that Boolean algebras are actually special instances, rather than analogs, of the general algebraic systems known as rings. (...).The algebraic theory developed in the previous communication is thus a particular instance of the theory of rings and can be deduced in part from the known theorems concerning rings."

Stone, *Subsumption of the theory of Boolean algebras under the theory of rings*, 1935

Boolean rings

> "We have thus shown that the theory of Boolean algebras, with particular reference to ideal-theoretic and arithmetic properties, is a special case of the theory of rings. We can say further that the theory of those algebraic systems known as distributive lattices can be considered as a part of the theory of rings : for an unpublished result due to Mr. Holbrook MacNeille shows that every distributive lattice can be imbedded by a purely algebraic construction in a Boolean algebra."

Stone, *Subsumption of the theory of Boolean algebras under the theory of rings*, 1935

"A full representation theory for Boolean algebras by fields of sets has been developed by Stone, and it is interesting to see the complications which arise in the more general case of distributive lattices. These show that the assumption that complements exist cannot be eliminated in Stone's theory."

Birkhoff, *Rings of sets*, 1937

Lattices as generalization

"In closing, I wish to mention briefly the generalization from Boolean algebras to distributive lattices. While the methods and results of the theory of Boolean algebras can be extended, with suitable modifications, to the case of distributive lattices, the direct connection with the theory of rings is lost."

"The relation between distributive lattices and Boolean rings is in this respect analogous to that between domains of integrity (commutative rings without divisors of zero) and fields."

Stone, *The representation of Boolean Algebras*, 1938

Corollary

Theorem
Any distributive lattice L is isomorphic with a ring of sets. (The converse is obvious.)

(...)
"[W]e get besides, as a corollary of Theorem 5.8, Stone's results :"

Theorem
Any Boolean algebra is isomorphic with a field of sets. (The converse is obvious.)

Birkhoff, *Lattice theory*, 1940

Crossing the theories

The representation problem

- A shared problem solved with shared methods
- Highlights a common development of Boolean algebras and Lattices
- The mathematical similarity is used as a non-formal level

Conclusion : Theories and the history of mathematics

Theories and the history of mathematics

How can we use the notion of a "theory" in the history of mathematics ?

- The *a priori* definition of a theory results in putting aside interactions between domains
- The notion is used by the actors themselves
  - In what context(s) ?
  - For what reason ?

Thank you for your attention