The Cuntz Semigroup Domain Theoretical Methods for C*-Algebras

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The Cuntz semigroup as an invariant for C*-algebras

was introduced under this title by K. Coward, G. A: Elliott and C. Ivanescu (J. Reine Angew. Math. 623 (2008), 161–193). It is seen as a refinement of K-theory with the aim of classifying C*-algebras by invariants.
Some of the properties of Cuntz semigroups of C*-algebras have been isolated to the concept of an abstract Cuntz semigroup.

A variant of domain theory is used where directed sets are restricted to increasing sequences. (One may ask whether this is appropriate, in general.) For separable C*-algebras the associated Cuntz semigroup is countably based, and in the countably based case there is no difference to standard domain theory.

Definition: An abstract Cuntz semigroup

is a commutative monoid (S,+,0) endowed with a partial order $\leq$ with the following properties:

(1) $(S, \leq)$ is ω-complete, i.e., every increasing sequence has a least upper bound,

(2) for every element a there is a sequence $a_1 \ll_\omega a_2 \ll_\omega \dots$ such that $a = \sup_n a_n$,

(3) addition is ω-continuous,

(4) addition preserves $\ll_\omega$, that is,
$$a_1 \ll_\omega b_1, a_2 \ll_\omega b_2 \implies a_1 + a_2 \ll_\omega b_1 + b_2$$

where $a \ll_\omega b$ means: for every increasing sequence x_n with $b \leq \sup_n x_n$ there is an N such that $a \leq x_N$.

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For this talk I will omit the restriction to increasing sequences.
Thus a Cuntz semigroup will be a structure $(S, +, 0, \leq)$ such that $(S, +, 0)$ is a commutative monoid and $(S, \leq)$ a continuous domain such that $+$ is Scott-continuous and $\ll$-preserving.

Abstract bases

First examples of Cuntz semigroups:

$$\mathbf{2} = \{0 < +\infty\}$$
$$\overline{\mathbb{R}}_+ = \{r \in \mathbb{R} \mid r \geq 0\} \cup \{+\infty\}, \text{ usual order.}$$

Recall: A subset B of a continuous dcpo P is a basis if, for every element $b \in P$, the set $\{a \in B \mid a \ll b\}$ is directed and $b = \sup\{a \in B \mid a \ll b\}$. The relation $\ll$ restricted to the basis has the following properties:

(1) $\ll$ is transitive,
(2) For every $b \in B$ there is an $a \in B$ such that $a \ll b$,
(3) has the interpolation property

$$a_i \ll c \quad (i = 1, 2) \implies \exists b.\ a_i \ll b \ll c \quad (i = 1, 2)$$

A set B with a relation $\ll$ having these three properties is called an abstract basis.

Pre-Cuntz semigroups

Structure $(P, +, 0, \ll)$ where $(P, +, 0)$ is a commutative monoid and $(P, \ll)$ an abstract basis such that $+$ is continuous and preserves $\ll$. (Continuity means: If $c \ll a + b$ there are $a' \ll a, b' \ll b$ such that $c \ll a' + b'$.)

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A round ideal is a subset J with (1) $0 \in J$,
(2) $a_i \in J\ (i = 1, 2) \implies \exists b \in J.\ a_i \ll b\ (i = 1, 2)$,
(3) $b \in J, a \ll b \implies a \in J$.

Completion

The round ideal completion of a pre-Cuntz semigroup is Cuntz semigroup.

If we require that, for every $a \in P$, there is sequence $a_1 \ll a_2 \ll \dots$ such that every $b \ll a$ is majorized by some a_n, then we have a first countable pre-Cuntz semigroup and we can form a round ω-ideal completion.

Basic example

Let X be a locally compact Hausdorff space,
$C_0(X)$ the C*-algebra of continuous $f\colon X \to \mathbb{C}$ vanishing at infinity,
$C_0(X)_+$ its positive cone, i.e., the set of all continuous $f\colon X \to \mathbb{R}_+$.

Notation: For $f \in C_0(X)_+$ and $\varepsilon > 0$, let

$$(f - \varepsilon)_+ \text{ be defined by } \max(f(x) - \varepsilon, 0)$$

For $e, f \in C_0(X)_+$, define

$$e \ll f \iff \exists \varepsilon > 0.\ e \leq (f - \varepsilon)_+$$

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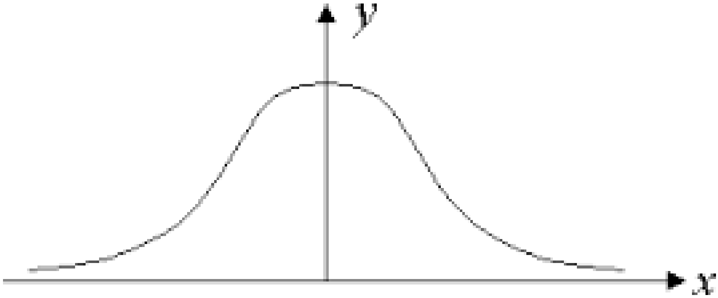
In particular $(f - \varepsilon)_+ \ll (f - \delta)_+ \ll f$, whenever $0 < \delta < \varepsilon$. Note: $f = \sup_n (f - \frac{1}{n})_+$ (pointwise) , hence f is the sup of a sequence $f_n \ll f_{n+1}$. Thus,
$(C_0(X)_+, +, 0, \ll)$ is a first countable pre-Cuntz semigroup.



y
x

Example (ctd.)

Let $\mathrm{LSC}(X)$ be the set of all $g\colon X \to \overline{\mathbb{R}}_+$ that are lower semicontinuous (i.e., $\{x \in X \mid f(x) > r\}$ is open for every $r \in \mathbb{R}_+$) with pointwise defined order and addition.

Note

$\mathrm{LSC}(X)$ is a Cuntz semigroup, in fact, the round ideal completion of $C_0(X)_+$.

$g \ll h$ if there are $f \in C_0(X)_+$ and $\varepsilon > 0$ such that $g \leq (f - \varepsilon)_+ \ll f \leq g$.

Dual S^* of a pre-Cuntz semigroup S

S^* is the set of all lower semicontinuous monoid homomorphisms $\lambda\colon S \to \overline{\mathbb{R}}_+$ with pointwise defined order, addition and multiplication by real numbers $r > 0$.

τ weakest topology on S^* for which the evaluation maps $\lambda \mapsto \lambda(x)\colon S^* \to \overline{\mathbb{R}}_+$, $x \in S$, are lower semicontinuous,

τ_c the co-compact topology associated to τ,

τ_p the patch topology (generated by τ and τ_c).

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Banach-Alaoglu Theorem

τ and τ_c are stably compact topologies, τ_p is compact T_2.
(S^*, τ) and (S^*, τ_p) are topological cones in the sense that addition $(\lambda, \mu) \mapsto \lambda + \mu$ and scalar multiplication $(r, \lambda) \mapsto r\lambda\colon \mathbb{R}_{>0} \times S^* \to S^*$ are continuous,
where $\mathbb{R}_{>0}$ is endowed with the upper ($=$ Scott) topology in the first case, with the usual Hausdorff topology in the second case.

Dual (ctd.)

A pre-Cuntz semigroup and its round ideal completion have the same dual.

The above Theorem is due to Plotkin for continuous d-cones with an additive way below relation; a generalization by K.K. covers Cuntz semigroups.

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Elliott, Ivanescu and Santiago have proved this for the patch topology in two special cases (see later). They introduce the patch topology directly through a notion of convergence:
A net $(\lambda_i)_i$ in S^* converges to λ iff

$$\limsup_i \lambda_i(y) \leq \lambda(x) \leq \liminf_i \lambda_i(x)$$

for all x and all $y \ll x$. The right hand inequality would give the topology τ and the left hand inequality the co-compact topology τ_c.

Bidual S^{**} of a pre-Cuntz semigroup S

S^{**} the cone of all lower semicontinuous linear functionals $\Lambda\colon S^* \to \overline{\mathbb{R}}_+$ with pointwise defined order, addition and scalar multiplication, where S^* is considered with the weak*upper topology τ.

There is a canonical map $\delta\colon S \to S^{**}$: x is mapped to the point evaluation $\delta(x) : \lambda \mapsto \lambda(x)$. This canonical map can be extended to the round ideal completion $\widehat{S}$ of S.

Questions: Is S^{**} a Cuntz semigroup, can it be constructed directly from S, is it a kind of tensor product $\overline{\mathbb{R}}_+ \otimes \widehat{S}$?

Partial answer: $C_0(X)_+{}^{**} \cong \mathrm{LSC}(X)$
(Special case of the Schröder-Simpson Theorem).

Positive cone of a C^*-algebra

Let A be a C*-algebra, that is, a Banach algebra with an involution $x \mapsto x^*$ such that $\|x^*\| = \|x\|$ and $\|xx^*\| = \|x^*x\|$.

A_+ is the positive cone of A, that is, the set of all elements that can be written in the form $a = x^*x$ for some $x \in A$.

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Functional calculus: Let $a \in A_+$. Then $a = a^*$ and the C*-subalgebra $C^*\langle a\rangle$ of A generated by a is commutative hence, by the Gelfand Representation Theorem, $C^*\langle a\rangle \cong C_0(X)$ for some locally compact Hausdorff space.
Thus we can form the element $(a-\varepsilon)_+ \in A_+$ for every $\varepsilon > 0$.

With respect to the relation $\ll$ defined by $a \ll b$ if there is an $\varepsilon > 0$ such that $a \leq (b-\varepsilon)_+$, A_+ becomes an abstract basis. But this relation does not interact nicely with addition as far as I can see.

Traces

Trace on A:
Monoid homomorphism $t\colon A_+ \to \overline{\mathbb{R}}_+$ such that $t(x^*x) = t(xx^*)$ for all $x \in A$.
$T(A)$ denotes the cone of all lower semicontinuous traces t.

Aim: Identify $T(A)$ as the dual of a pre-Cuntz semigroup.

A_+ as a pre-Cuntz semigroup

Cuntz-Pedersen equivalence on A_+: $a \sim a'$ if there is a sequence $(x_n)_n$ of elements in A such that $a = \sum_n x_n x_n^*$ and $a' = \sum_n x_n^* x_n$.

Definition

$a \prec\!\!\prec b$ if there is an $\varepsilon > 0$ and an a' such that $a \sim a' \leq (b - \varepsilon)_+$.

A_+ as a pre-Cuntz semigroup

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Definition

$a \prec\!\!\prec b$ if there is an $\varepsilon > 0$ and an a' such that $a \sim a' \leq (b - \varepsilon)_+$.

Theorem

$(A_+, +, 0, \prec\!\!\prec)$ is a pre-Cuntz semigroup and $T(A)$ its dual.

All the ingredients for the proof are in the literature.

The round ideal completion $\widehat{A_+}$ of $(A_+, \prec\!\!\prec)$ is contained in the dual $T(A)^* \cong (A_+, \prec\!\!\prec)^{**}$.

Conjecture $\widehat{A_+}$ is the dual of $T(A)$.

Hilbert modules

Let A be a C*-algebra. A pre-Hilbert A-module is a right A-module X together with a map $(x, y) \mapsto \langle a, b\rangle : X \times X \to A$ such that, for $x, y, z \in X$, $\alpha, \beta \in \mathbb{C}$ and $a \in A$, the following laws hold:

1. $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$
2. $\langle x, ya\rangle = \langle x, y\rangle a$
3. $\langle y, x\rangle = \langle x, y\rangle^*$
4. $\langle x, x\rangle \in A_+, \langle x, x\rangle = 0$ iff $x = 0$.

If X is complete for the norm $\sqrt{\|\langle x, x\rangle\|}$, it is called a Hilbert A-module.

In K-theory one looks at isomorphism classes of finitely generated (projective) Hilbert A-modules, addition being direct sum. In order to obtain a finer invariant, Coward, Elliott, and Ivanescu looked at countably generated Hilbert A-modules.

Towards the Cuntz semigroups of a C^*-algebra A

Consider the class A-Hilb of countably generated Hilbert A-modules. In A-Hilb we use direct sum $\oplus$ as addition.

An Hibert A-submodule Y of X is said to be compactly contained in X and we write $X \ll Y$, if there is a compact endomorphism on X that restricts to the identity on Y.

We write $Y \lll X$ if there is a Hilbert A-submodule Y' compactly contained in X and isomorphic to Y:

$$Y \lll X \iff \exists Y'.\ Y \sim Y' \ll X$$

Coward, Elliott,Ivanescu

$(A\text{-Hilb}, \oplus, 0, \lll)$ is an ω-pre-Cuntz semigroup.

The functor Cu

We introduce a preorder $\precsim$: $X \precsim X'$ if $Y \lll X \implies Y \lll X'$.
The associated equivalence is $X \sim X'$ if $Y \lll X \iff Y \lll X'$.

The Cuntz semigroup of A

$\mathrm{Cu}(A) = (A\text{-}Hilb/\sim, \oplus, 0, \leq)$ is an ω-Cuntz semigroup, the way below relation being induced by $\lll$.

The category of C*-algebras:
objects: C*-algebras
morphisms: Algebra homomorphisms preserving involution

The category of abstract Cuntz semigroups:
objects: abstract ω-Cuntz semigroups
morphisms: ω-continuous monoid homomorphisms preserving $\ll_\omega$

Cu extends to a functor that preserves direct sums, direct limits (of sequences), tensor products.

Quasitraces on a C*-algebra A

A quasitrace is a function $t\colon A_+ \to \overline{\mathbb{R}}_+$ with the following properties:
(1) t is a monoid homomorphism on the positive part of every commutative
C*-subalgebra,
(2) $t(xx^*) = t(x^*x)$ for all $x \in A$.

$QT(A)$ denotes the set of all lower semicontinuous quasitraces on A with pointwise defined order, addition and multiplication with scalars $r > 0$.

Elliott, Robert, Santiago

$QT(A)$ is isomorphic to the dual of the Cuntz semigroup of A:
$$QT(A) \cong Cu(A)^*$$

Question: Characterize the bidual $Cu(A)^{**} \cong QT(A)^*$.

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Historical Note

In the late 1960ies K H Hofmann was working on C*-algebras. The Dauns-Hofmann Theorem from that time can be found in almost any book on C*-algebras. A main example for continuous lattices was the lattice of closed ideals of a C*-algebra.
Compact semigroups were the main field of interest of Hofmann, Keimel, Lawson, Mislove in the 1960ies. A key work was
K H Hofmann, A L Stralka: The algebraic Theory of Compact Semilattices. *Dissertationes Mathematicae* CXXXVII (1976), 54 pages.
This was written in 1974. On p. 27 one finds the following:

The Definition

It is now notationally convenient to call an element y in a lattice L relatively compact under x iff it is contained in any ideal I of L with $\sup I \geq x$. This is ostensibly equivalent to saying that for all subsets $X \subseteq L$ with $\sup X \geq x$ there is a finite subset subset $Y \subseteq X$ with $y \leq \sup Y$. Let us call a lattice relatively algebraic if it is complete and every element in it is the l.u.b. of all relatively compact elements under it.

It was in the sequel of this work that it was discovered that this structure was equivalent to the continuous lattices introduced by D S Scott in his seminal 1972 paper.