Oles Embeddings
(work in progress)

Paul Blain Levy

University of Birmingham

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Frank Oles in his PhD thesis (1982) defined a category of store shapes for semantics of local state with John Reynolds.
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It’s connected to numerous structures in the semantics of effects:

- Power and Plotkin’s lookup-update algebras
- Melliès’ redundancy theorem for lookup equations
- Power and Shkaravska’s account of arrays as comodels
- Hyland, Plotkin and Power’s combination of a functor and a monad
- My account of monads supporting exception handling
- Hermida and Tennent’s account of monoidal indeterminates
- Johnson et al’s account of lenses.
Three levels of generality

1. Oles embedding in a category
2. Oles embedding across an action
3. Base for a monad
Three levels of generality

1. Oles embedding in a category
2. Oles embedding across an action
3. Base for a monad

Most of the talk will be about (1).
- Oles embeddings and their complements
- Oles expansions and their quotients
- Oles intersections.
The complementor of an injection $f : A \hookrightarrow B$ is the function $f^c : B \to B + A$ sending
- $f(a) \mapsto \text{inr } a$
- $b \mapsto \text{inl } b$ if $b \notin \text{range}(f)$. 
The complementor of an injection \( f : A \hookrightarrow B \) is the function \( f^c : B \rightarrow B + A \) sending

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We then have the equations:

\[
\begin{align*}
A & \xrightarrow{f} B & B & \xrightarrow{f^c} B + A & B & \xrightarrow{f^c} B + A \\
\downarrow \text{inr} & \downarrow f^c & \downarrow \text{id} & \downarrow [\text{id},f] & \downarrow f^c & \downarrow f^c + A \\
B + A & \quad & B & \quad & B + A & \quad \\
\end{align*}
\]
Basic definition

Let $C$ be a category with binary coproducts and initial object. We form a category $\text{Oles}(C)$ with the same objects as $C$. 
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An Oles embedding $f : A \to B$ consists of

- an map $f^i : A \to B$ (the injection)
- a map $f^c : B \to B + A$ (the complementor)

satisfying the equations:

\[
\begin{align*}
A & \xrightarrow{f^i} B \\
\downarrow \text{inr} & \quad \downarrow f^c \\
B + A & \quad B + A
\end{align*}
\]

\[
\begin{align*}
B & \xrightarrow{f^c} B + A \\
\downarrow \text{id} & \quad \downarrow [\text{id}, f^i] \\
B & \quad (B + A) + A
\end{align*}
\]
Making a category

The identity on $A$ has injection $\text{id}_A$ and complementor

$$\text{inr} : A \rightarrow A + A$$

The composite of $f : A \rightarrow B$ and $g : B \rightarrow C$ has injection

$$A \xrightarrow{f^i} B \xrightarrow{g^i} C$$

and complementor

$$C \xrightarrow{g^c} C + B \xrightarrow{C + f^c} C + (B + A) \xrightarrow{[\text{inl}, g^i + A]} C + A$$

Theorem

$\text{Oles}(\textbf{Set})$ is the category of sets and injections.
Basic properties

- \((\text{Oles}(\mathcal{C}), 0, +)\) is a symmetric monoidal category with initial unit.
- Its groupoid of isomorphisms is the same as that of \(\mathcal{C}\).
A coproduct embedding $A \hookrightarrow B$ consists of an object $X$ and $\alpha : X + A \cong B$. 
A coproduct embedding $A \rightarrowtail B$ consists of an object $X$ and $\alpha : X + A \cong B$.

These give a bicategory with the same objects as $C$.

A 2-cell from $(X, \alpha)$ to $(Y, \beta)$ is $h : X \rightarrow Y$ such that

$$
\begin{array}{c}
X + A \xymatrix{ \ar[r]^\alpha & B} \\
h + A \ar[d] \\
Y + A \ar[ur]_{\beta}
\end{array}
$$
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A 2-cell from $(X, \alpha)$ to $(Y, \beta)$ is $h : X \to Y$ such that

\[
\begin{array}{ccc}
X + A & \overset{\alpha}{\longrightarrow} & B \\
h + A & \downarrow & \beta \\
Y + A & \end{array}
\]

Every coproduct embedding gives rise to an Oles embedding, so there’s a functor from the bicategory to $\text{Oles}(C)$. 
A complement of an Oles embedding $f : A \leftrightarrow B$ is a coproduct embedding that gives rise to it. These form a category.
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Does every Oles embedding have

- a complement? Not necessarily
- an essentially unique complement? If \( C \) is extensive.
- a terminal complement? If \( C \) has equalizers preserved by \(- + X\)
- an initial complement? Not necessarily
Let $C$ have binary products and a terminal object.

An **Oles expansion** $A \to B$ is an Oles embedding in $C^{\text{op}}$.
Let $\mathcal{C}$ have binary products and a terminal object.

An **Oles expansion** $A \to B$ is an Oles embedding in $\mathcal{C}^{\text{op}}$

- a morphism $p : B \to A$ (the projection)
- a morphism $\bullet : B \times A \to B$ (the overwriter)

satisfying

\[
\forall b \in B, a \in A. \quad p(b \bullet a) = a
\]

\[
\forall b \in B. \quad b \bullet p(b) = b
\]

\[
\forall b \in B, a, a' \in A. \quad (b \bullet a) \bullet a' = b \bullet a'
\]

Also called a **very well-behaved total lens**.
A product expansion $A \rightarrow B$ consists of an object $X$ and $X \times A \cong B$. We take 2-cells to be $C$-morphisms.
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A quotient of an Oles expansion $f : A \to B$ is a product expansion that gives rise to it.
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In $\textbf{Set}$, the Oles expansion $0 \to 0$ has initial quotient $0$ and terminal quotient $1$. 

Quotients of an Oles expansion

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A quotient of an Oles expansion $f : A \to B$ is a product expansion that gives rise to it.

In $\mathbf{Set}$, the Oles expansion $0 \to 0$ has initial quotient $0$ and terminal quotient $1$.

Oles proved: in $\mathbf{Set}$, every expansion has an initial quotient.
A square of Oles embeddings

\[
\begin{array}{ccc}
A & \xrightarrow{h} & B \\
\downarrow{k} & & \downarrow{g} \\
C & \xrightarrow{f} & D
\end{array}
\]

is an Oles intersection square when

\[
D \xrightarrow{f^c} D + C
\]

\[
\begin{array}{ccc}
D + B & \xrightarrow{f^c + h^c} & (D + C) + (B + A) \\
\downarrow{g^c} & & \downarrow{g^c + k^c} \\
(D + B) + (C + A)
\end{array}
\]

Such squares compose.

If \(A = 0\) then \(f^c\) and \(g^c\) are disjoint.
Oles intersection square

A square of Oles embeddings

\[ A \xrightarrow{h} B \]
\[ \downarrow k \quad \downarrow g \]
\[ C \xrightarrow{f} D \]

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\[ D \xrightarrow{f^c} D + C \]
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\[ D + B \xrightarrow{f^c + h^c} (D + C) + (B + A) \xrightarrow{\mathbb{R}} (D + B) + (C + A) \]

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\downarrow g^c + k^c
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Such squares compose.

If \( A = 0 \) then \( f \) and \( g \) are disjoint.
Are they pullbacks?

An Oles intersection square \[ A \rightarrow^h B \rightarrow^k C \leftarrow_f D \] is not a pullback in general.

It is if binary coproducts are extensive. If there exists a \[ C \rightarrow D \rightarrow A \] it's an absolute pullback in \( C \). Proved by Trnková for \( C = \text{Set} \).

It's also a pullback in Oles \( (C) \), provided \(-+Y\) preserves pullbacks.
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If there exists a \( C \)-morphism \( D \to A \) it’s an \textbf{absolute} pullback in \( C \).

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If there exists a \(C\)-morphism \(D \rightarrow A\) it’s an absolute pullback in \(C\).

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It’s also a pullback in \(\text{Oles}(C)\), provided \(- + Y\) preserves pullbacks.
Properties of disjointness

The cocone $A_0 \rightarrow \cdots \rightarrow A_{n-1}$ in $n-1 \leftarrow \cdots \leftarrow \sum_{i<n} A_i$ is pairwise disjoint.

For any pairwise disjoint cocone $A_0 \downarrow \cdots \downarrow A_{n-1}$ there's a unique Oles embedding $\sum_{i<n} A_i \rightarrow B$ that's a morphism of cocones.
Properties of disjointness

The cocone

\[ \sum_{i < n} A_i \]

is pairwise disjoint.

For any pairwise disjoint cocone

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there’s a unique Oles embedding \( \sum_{i < n} A_i \to B \) that’s a morphism of cocones.
Covering intersection squares

Given two Oles intersection squares

If the inner one is covering then there is a unique Oles embedding $D \hookrightarrow E$ that's a morphism of cocones.
Given two Oles intersection squares

if the inner one is **covering** then there is a unique Oles embedding $D \hookrightarrow E$ that’s a morphism of cocones.

This may be generalized to other diagram shapes.
A monad $T$ on a category $\mathcal{D}$ gives a comonad $F^T U^T$ on the Eilenberg-Moore category $C^T$. 
A monad $T$ on a category $\mathcal{D}$ gives a comonad $F^T U^T$ on the Eilenberg-Moore category $\mathcal{C}^T$.

A coalgebra for this comonad is called a $T$-base. Lack, Taylor, Jacobs . . .

This consists of an object $P$ and maps $\theta : TP \rightarrow P$ and $\phi : P \rightarrow TP$, satisfying 5 equations, of which 2 are redundant.
A monoidal action of a symmetric monoidal category \((\mathcal{C}, I, \otimes)\) on a category \(\mathcal{D}\) is a map \(\otimes : \mathcal{D} \times \mathcal{C} \rightarrow \mathcal{D}\) and isomorphisms

\[
P \otimes (B \otimes C) \cong (P \otimes B) \otimes C
\]

\[
P \otimes I \cong P
\]

satisfying the pentagon and the triangle.
Suppose $\mathcal{C}$ has binary coproducts and an initial object, and acts monoidally on $\mathcal{D}$. Any $A$ in $\mathcal{C}$ gives a monad $P \mapsto P \otimes A$ on $\mathcal{D}$. A base structure on $P$ for this monad is called an Oles embedding $A \hookrightarrow P$. We can compose Oles embeddings $A \hookrightarrow B \hookrightarrow P$ and speak of disjoint embeddings and intersection squares into $P$. 
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Any $A$ in $\mathcal{C}$ gives a monad $P \mapsto P \otimes A$ on $\mathcal{D}$.

A base structure on $P$ for this monad is called an **Oles embedding** $A \hookrightarrow P$.

We can compose Oles embeddings

$$A \hookrightarrow B \hookrightarrow P$$

and speak of disjoint embeddings and intersection squares into $P$. 
Oles embeddings in a category

$C$ acts monoidally on itself.

This gives Oles embeddings in $C$. 
Examples

Oles embeddings in a category

\( \mathcal{C} \) acts monoidally on itself.

This gives Oles embeddings in \( \mathcal{C} \).

Lookup/update algebras

\( \text{Set}^{\text{op}} \) acts monoidally on \( \text{Set} \) via exponentiation.

An Oles embedding from \( A \hookrightarrow P \) is a lookup/update algebra structure on \( P \),

shown by Plotkin and Power to be an algebra for the state monad \( X \mapsto A \to (A \times X) \).
Handling exceptions and reading

Let \textbf{MonadSet} be the category of monads on \textbf{Set}.

- \textbf{Set} acts on \textbf{MonadSet} via

\[
(T \otimes E)X = T(X + E)
\]
Handling exceptions and reading

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- $\text{Set}^{\text{op}}$ acts on $\text{MonadSet}$ via

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- $\text{Set}^{\text{op}} \times \text{Set}$ acts on $\text{MonadSet}$ via

\[(T \otimes (S, E))X = S \to T(X + E)\]
Handling exceptions and reading

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- **Set** acts on **MonadSet** via
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- **Set^{op}** acts on **MonadSet** via
  \[(T \otimes S)X = S \rightarrow TX\]

- **Set^{op} \times Set** acts on **MonadSet** via
  \[(T \otimes (S, E))X = S \rightarrow T(X + E)\]

An Oles embedding \((S, E) \hookrightarrow T\)
says how **T** models effect handling for reading and exceptions.
Handling exceptions and reading

Let $\text{MonadSet}$ be the category of monads on $\text{Set}$.  

- $\text{Set}$ acts on $\text{MonadSet}$ via 
  \[(T \odot E)X = T(X + E)\]

- $\text{Set}^{\text{op}}$ acts on $\text{MonadSet}$ via 
  \[(T \odot S)X = S \rightarrow TX\]

- $\text{Set}^{\text{op}} \times \text{Set}$ acts on $\text{MonadSet}$ via 
  \[(T \odot (S, E))X = S \rightarrow T(X + E)\]

An Oles embedding $(S, E) \hookrightarrow T$ says how $T$ models effect handling for reading and exceptions.

There's a variant for I/O effect handling using the Hyland-Plotkin-Power monad formula $X \hookrightarrow \mu Y. T(X + HY)$
In a category

- Oles embeddings and their complements
- Oles expansions and their quotients
- Oles intersections
- Covering Oles intersections are initial.
Summary

In a category
- Oles embeddings and their complements
- Oles expansions and their quotients
- Oles intersections
- Covering Oles intersections are initial.

Oles embeddings across an action includes many structures of interest.