Oles Embeddings
(work in progress)

Paul Blain Levy

University of Birmingham

September 1, 2015

Background

Frank Oles in his PhD thesis (1982) defined a category of store shapes for semantics of local state with John Reynolds.

Background

Frank Oles in his PhD thesis (1982) defined a category of store shapes for semantics of local state with John Reynolds.

I applied his construction to a game category in
"Global state considered helpful" (MFPS 2008)

Background

Frank Oles in his PhD thesis (1982) defined a category of store shapes for semantics of local state with John Reynolds.

I applied his construction to a game category in
"Global state considered helpful" (MFPS 2008)

Background

Frank Oles in his PhD thesis (1982) defined a category of store shapes for semantics of local state with John Reynolds.

I applied his construction to a game category in
"Global state considered helpful" (MFPS 2008)

It's connected to numerous structures in the semantics of effects:

- Power and Plotkin's lookup-update algebras
- Melliès' redundancy theorem for lookup equations
- Power and Shkaravska's account of arrays as comodels
- Hyland, Plotkin and Power's combination of a functor and a monad
- My account of monads supporting exception handling
- Hermida and Tennent's account of monoidal indeterminates
- Johnson et al's account of lenses.

Three levels of generality

1. Oles embedding in a category
2. Oles embedding across an action
3. Base for a monad

Three levels of generality

1. Oles embedding in a category
2. Oles embedding across an action
3. Base for a monad

Most of the talk will be about (1).

- Oles embeddings and their complements
- Oles expansions and their quotients
- Oles intersections.

Complementors

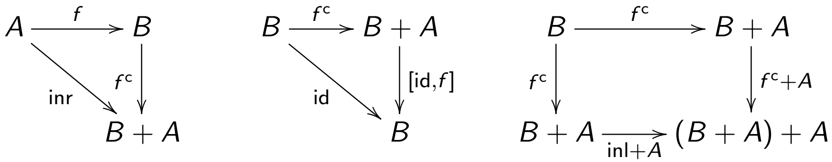
The complementor of an injection $f : A \rightarrowtail B$ is the function $f^{\mathsf{c}} : B \to B + A$ sending

- $f(a) \mapsto \mathsf{inr}\ a$
- $b \mapsto \mathsf{inl}\ b$ if $b \notin \mathrm{range}(f)$.

The complementor of an injection $f : A \rightarrowtail B$ is the function $f^{\mathsf{c}} : B \to B + A$ sending

- $f(a) \mapsto \mathsf{inr}\ a$
- $b \mapsto \mathsf{inl}\ b$ if $b \notin \mathrm{range}(f)$.

We then have the equations:



Basic definition

Let $\mathcal{C}$ be a category with binary coproducts and initial object.
We form a category $\mathrm{Oles}(\mathcal{C})$ with the same objects as $\mathcal{C}$.

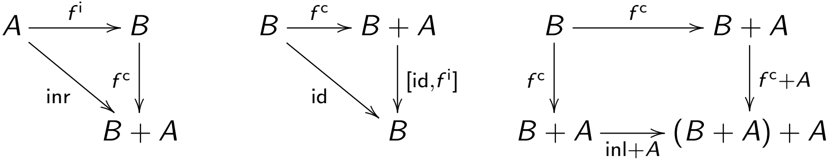
Basic definition

Let $\mathcal{C}$ be a category with binary coproducts and initial object.
We form a category $\mathrm{Oles}(\mathcal{C})$ with the same objects as $\mathcal{C}$.

An Oles embedding $f : A \rightarrowtail B$ consists of

- an map $f^{\mathsf{i}} : A \to B$ (the injection)
- a map $f^{\mathsf{c}} : B \to B + A$ (the complementor)

satisfying the equations:



Making a category

The identity on A has injection id_A and complementor

$$\mathrm{inr} \ : \ A \to A + A$$

The composite of $f \ : \ A \rightarrowtail B$ and $g \ : \ B \rightarrowtail C$ has injection

$$A \xrightarrow{f^{\mathrm{i}}} B \xrightarrow{g^{\mathrm{i}}} C$$

and complementor

$$C \xrightarrow{g^{\mathrm{c}}} C + B \xrightarrow{C + f^{\mathrm{c}}} C + (B + A) \xrightarrow{[\mathrm{inl}, g^{\mathrm{i}} + A]} C + A$$

Theorem

$\mathrm{Oles}(\mathbf{Set})$ is the category of sets and injections.

Basic properties

- $(\mathrm{Oles}(\mathcal{C}), 0, +)$ is a symmetric monoidal category with initial unit.
- Its groupoid of isomorphisms is the same as that of $\mathcal{C}$.

From coproduct embeddings to Oles embeddings

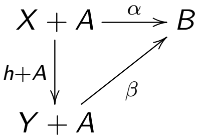
A coproduct embedding $A \rightarrowtail B$ consists of
an object X and $\alpha : X + A \cong B$.

From coproduct embeddings to Oles embeddings

A coproduct embedding $A \rightarrowtail B$ consists of
an object X and $\alpha : X + A \cong B$.

These give a bicategory with the same objects as $\mathcal{C}$.

A 2-cell from (X, α) to (Y, β) is $h : X \to Y$ such that

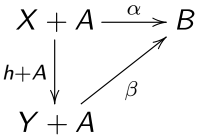


From coproduct embeddings to Oles embeddings

A coproduct embedding $A \rightarrowtail B$ consists of
an object X and $\alpha : X + A \cong B$.

These give a bicategory with the same objects as $\mathcal{C}$.

A 2-cell from (X, α) to (Y, β) is $h : X \to Y$ such that



Every coproduct embedding gives rise to an Oles embedding, so there's a functor from the bicategory to $\mathrm{Oles}(\mathcal{C})$.

Complements of an Oles embedding

A complement of an Oles embedding $f : A \rightarrowtail B$ is a coproduct embedding that gives rise to it.
These form a category.

Complements of an Oles embedding

A complement of an Oles embedding $f : A \rightarrowtail B$ is a coproduct embedding that gives rise to it.
These form a category.

Does every Oles embedding have

- a complement? Not necessarily
- an essentially unique complement? If $\mathcal{C}$ is extensive.
- a terminal complement? If $\mathcal{C}$ has equalizers preserved by $- + X$
- an initial complement? Not necessarily

Oles expansions

Let $\mathcal{C}$ have binary products and a terminal object.

An Oles expansion $A \to B$ is an Oles embedding in $\mathcal{C}^{\mathsf{op}}$

Oles expansions

Let $\mathcal{C}$ have binary products and a terminal object.

An Oles expansion $A \to B$ is an Oles embedding in $\mathcal{C}^{\mathsf{op}}$

- a morphism $p \ : \ B \to A$ (the projection)
- a morphism $\bullet \ : \ B \times A \to B$ (the overwriter)

satisfying

$$\begin{array}{lrcl} \forall b \in B, a \in A. & p(b \bullet a) & = & a \\ \forall b \in B. & b \bullet p(b) & = & b \\ \forall b \in B, a, a,' \in A. & (b \bullet a) \bullet a' & = & b \bullet a' \end{array}$$

Also called a very well-behaved total lens.

Quotients of an Oles expansion

A product expansion $A \to B$ consists of an object X and $X \times A \cong B$.
We take 2-cells to be $\mathcal{C}$-morphisms.

Quotients of an Oles expansion

A product expansion $A \to B$ consists of an object X and $X \times A \cong B$.
We take 2-cells to be $\mathcal{C}$-morphisms.

A quotient of an Oles expansion $f : A \to B$ is
a product expansion that gives rise to it.

Quotients of an Oles expansion

A product expansion $A \to B$ consists of an object X and $X \times A \cong B$.
We take 2-cells to be $\mathcal{C}$-morphisms.

A quotient of an Oles expansion $f : A \to B$ is
a product expansion that gives rise to it.

In **Set**, the Oles expansion $0 \to 0$ has initial quotient 0 and terminal quotient 1.

Quotients of an Oles expansion

A product expansion $A \to B$ consists of an object X and $X \times A \cong B$.
We take 2-cells to be $\mathcal{C}$-morphisms.

A quotient of an Oles expansion $f : A \to B$ is
a product expansion that gives rise to it.

In **Set**, the Oles expansion $0 \to 0$ has initial quotient 0 and terminal quotient 1.

Oles proved: in **Set**, every expansion has an initial quotient.

Oles intersection square

A square of Oles embeddings

$$\begin{array}{ccc} A & \overset{h}{\rightarrowtail} & B \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ C & \underset{f}{\rightarrowtail} & D \end{array}$$

is an Oles intersection square when

$$\begin{array}{ccccc} D & & \overset{f^{c}}{\longrightarrow} & & D + C \\ {\scriptstyle g^{c}}\downarrow & & & & \downarrow{\scriptstyle g^{c}+k^{c}} \\ D + B & \underset{f^{c}+h^{c}}{\longrightarrow} & (D + C) + (B + A) & \underset{\cong}{\longrightarrow} & (D + B) + (C + A) \end{array}$$

Oles intersection square

A square of Oles embeddings

$$\begin{array}{ccc} A & \overset{h}{\rightarrowtail} & B \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ C & \underset{f}{\rightarrowtail} & D \end{array}$$

is an Oles intersection square when

$$\begin{array}{ccccc} D & & \xrightarrow{\quad f^{c} \quad} & & D + C \\ {\scriptstyle g^{c}}\downarrow & & & & \downarrow{\scriptstyle g^{c}+k^{c}} \\ D + B & \xrightarrow[f^{c}+h^{c}]{} & (D + C) + (B + A) & \xrightarrow[\cong]{} & (D + B) + (C + A) \end{array}$$

Such squares compose.

Oles intersection square

A square of Oles embeddings
$$\begin{array}{ccc} A & \overset{h}{\rightarrowtail} & B \\ {\scriptstyle k}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ C & \underset{f}{\rightarrowtail} & D \end{array}$$

is an Oles intersection square when

$$\begin{array}{ccccc} D & & \xrightarrow{\quad f^{c} \quad} & & D + C \\ {\scriptstyle g^{c}}\big\downarrow & & & & \big\downarrow{\scriptstyle g^{c}+k^{c}} \\ D + B & \xrightarrow[f^{c}+h^{c}]{} & (D + C) + (B + A) & \xrightarrow[\cong]{} & (D + B) + (C + A) \end{array}$$

Such squares compose.

If $A = 0$ then f and g are disjoint.

Are they pullbacks?

An Oles intersection square
$$\begin{array}{ccc} A & \overset{h}{\rightarrowtail} & B \\ {\scriptstyle k}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ C & \underset{f}{\rightarrowtail} & D \end{array}$$
is not a pullback in general.

Are they pullbacks?

An Oles intersection square

$$\begin{array}{ccc} A & \overset{h}{\rightarrowtail} & B \\ {\scriptstyle k}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ C & \underset{f}{\rightarrowtail} & D \end{array}$$

is not a pullback in general.

It is if binary coproducts are extensive.

Are they pullbacks?

An Oles intersection square $\begin{array}{ccc} A & \overset{h}{\rightarrowtail} & B \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ C & \underset{f}{\rightarrowtail} & D \end{array}$ is not a pullback in general.

It is if binary coproducts are extensive.

If there exists a $\mathcal{C}$-morphism $D \to A$ it's an absolute pullback in $\mathcal{C}$.
Proved by Trnková for $\mathcal{C} = \mathbf{Set}$.

Are they pullbacks?

An Oles intersection square

$$\begin{array}{ccc} A & \overset{h}{\rightarrowtail} & B \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ C & \underset{f}{\rightarrowtail} & D \end{array}$$

is not a pullback in general.

It is if binary coproducts are extensive.

If there exists a $\mathcal{C}$-morphism $D \to A$ it's an absolute pullback in $\mathcal{C}$.
Proved by Trnková for $\mathcal{C} = \mathbf{Set}$.

It's also a pullback in $\mathrm{Oles}(\mathcal{C})$, provided $- + Y$ preserves pullbacks.

Properties of disjointness

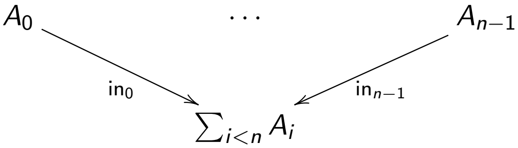
The cocone

$$\begin{array}{ccccc} A_0 & & \cdots & & A_{n-1} \\ & \searrow^{\mathsf{in}_0} & & {}^{\mathsf{in}_{n-1}}\swarrow & \\ & & \sum_{i<n} A_i & & \end{array}$$

is pairwise disjoint.

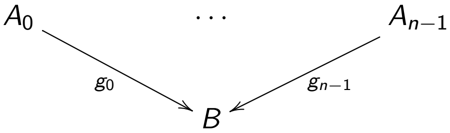
Properties of disjointness

The cocone



is pairwise disjoint.

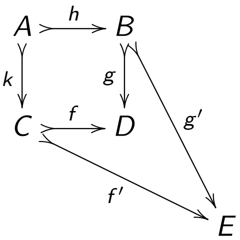
For any pairwise disjoint cocone



there's a unique Oles embedding $\sum_{i<n} A_i \to B$ that's a morphism of cocones.

Covering intersection squares

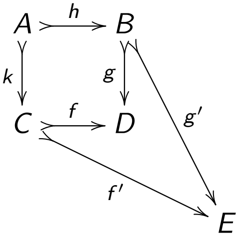
Given two Oles intersection squares



if the inner one is covering then there is a unique Oles embedding $D \rightarrowtail E$ that's a morphism of cocones.

Covering intersection squares

Given two Oles intersection squares



if the inner one is covering then there is a unique Oles embedding $D \rightarrowtail E$ that's a morphism of cocones.

This may be generalized to other diagram shapes.

Base for a monad

A monad T on a category $\mathcal{D}$ gives a comonad $F^T U^T$ on the Eilenberg-Moore category $\mathcal{C}^T$.

Base for a monad

A monad T on a category $\mathcal{D}$ gives a comonad $F^T U^T$ on the Eilenberg-Moore category $\mathcal{C}^T$.

A coalgebra for this comonad is called a T-base. Lack, Taylor, Jacobs ...

This consists of an object P and maps $\theta : TP \to P$ and $\phi : P \to TP$, satisfying 5 equations, of which 2 are redundant.

Monoidal actions

A monoidal action of a symmetric monoidal category $(\mathcal{C}, I, \otimes)$ on a category $\mathcal{D}$ is a map $\oslash : \mathcal{D} \times \mathcal{C} \to \mathcal{D}$ and isomorphisms

$$\begin{aligned} P \oslash (B \otimes C) &\cong (P \oslash B) \oslash C \\ P \oslash I &\cong P \end{aligned}$$

satisfying the pentagon and the triangle.

Oles embedding across a monoidal action

Suppose $\mathcal{C}$ has binary coproducts and an initial object,
and acts monoidally on $\mathcal{D}$.

Any A in $\mathcal{C}$ gives a monad $P \mapsto P \oslash A$ on $\mathcal{D}$.

A base structure on P for this monad is called an Oles embedding $A \rightarrowtail P$.

Oles embedding across a monoidal action

Suppose $\mathcal{C}$ has binary coproducts and an initial object,
and acts monoidally on $\mathcal{D}$.

Any A in $\mathcal{C}$ gives a monad $P \mapsto P \oslash A$ on $\mathcal{D}$.

A base structure on P for this monad is called an Oles embedding $A \rightarrowtail P$.

We can compose Oles embeddings

$$A \rightarrowtail B \rightarrowtail P$$

and speak of disjoint embeddings and intersection squares into P.

Examples

Oles embeddings in a category

$\mathcal{C}$ acts monoidally on itself.

This gives Oles embeddings in $\mathcal{C}$.

Examples

Oles embeddings in a category

$\mathcal{C}$ acts monoidally on itself.

This gives Oles embeddings in $\mathcal{C}$.

Lookup/update algebras

$\mathbf{Set}^{\mathrm{op}}$ acts monoidally on **Set** via exponentiation.

An Oles embedding from $A \rightarrowtail P$ is a lookup/update algebra structure on P,

shown by Plotkin and Power to be an algebra for the state monad $X \mapsto A \to (A \times X)$.

Handling exceptions and reading

Let **MonadSet** be the category of monads on **Set**.

- **Set** acts on **MonadSet** via

$$(\mathbf{T} \oslash E)X = \mathbf{T}(X + E)$$

Handling exceptions and reading

Let **MonadSet** be the category of monads on **Set**.

- **Set** acts on **MonadSet** via

$$(\mathbf{T} \oslash E)X = \mathbf{T}(X + E)$$

- $\mathbf{Set}^{\mathrm{op}}$ acts on **MonadSet** via

$$(\mathbf{T} \oslash S)X = S \to \mathbf{T}X$$

Handling exceptions and reading

Let **MonadSet** be the category of monads on **Set**.

- **Set** acts on **MonadSet** via

$$(\mathbf{T} \oslash E)X = \mathbf{T}(X + E)$$

- $\mathbf{Set}^{\mathrm{op}}$ acts on **MonadSet** via

$$(\mathbf{T} \oslash S)X = S \to \mathbf{T}X$$

- $\mathbf{Set}^{\mathrm{op}} \times \mathbf{Set}$ acts on **MonadSet** via

$$(\mathbf{T} \oslash (S, E))X = S \to \mathbf{T}(X + E)$$

Handling exceptions and reading

Let **MonadSet** be the category of monads on **Set**.

- **Set** acts on **MonadSet** via

$$(\mathbf{T} \oslash E)X = \mathbf{T}(X + E)$$

- $\mathbf{Set}^{\mathrm{op}}$ acts on **MonadSet** via

$$(\mathbf{T} \oslash S)X = S \to \mathbf{T}X$$

- $\mathbf{Set}^{\mathrm{op}} \times \mathbf{Set}$ acts on **MonadSet** via

$$(\mathbf{T} \oslash (S, E))X = S \to \mathbf{T}(X + E)$$

An Oles embedding $(S, E) \rightarrowtail \mathbf{T}$
says how $\mathbf{T}$ models effect handling for reading and exceptions.

Handling exceptions and reading

Let **MonadSet** be the category of monads on **Set**.

- **Set** acts on **MonadSet** via

$$(\mathbf{T} \oslash E)X = \mathbf{T}(X + E)$$

- $\mathbf{Set}^{\mathrm{op}}$ acts on **MonadSet** via

$$(\mathbf{T} \oslash S)X = S \to \mathbf{T}X$$

- $\mathbf{Set}^{\mathrm{op}} \times \mathbf{Set}$ acts on **MonadSet** via

$$(\mathbf{T} \oslash (S, E))X = S \to \mathbf{T}(X + E)$$

An Oles embedding $(S, E) \rightarrowtail \mathbf{T}$
says how **T** models effect handling for reading and exceptions.

There's a variant for I/O effect handling
using the Hyland-Plotkin-Power monad formula $X \mapsto \mu Y. \mathbf{T}(X + HY)$

Summary

In a category

- Oles embeddings and their complements
- Oles expansions and their quotients
- Oles intersections
- Covering Oles intersections are initial.

Summary

In a category

- Oles embeddings and their complements
- Oles expansions and their quotients
- Oles intersections
- Covering Oles intersections are initial.

Oles embeddings across an action includes many structures of interest.